CSCI 104

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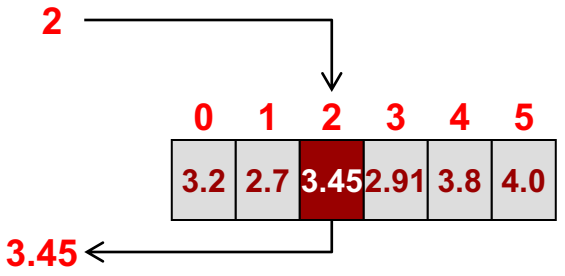
Slides adapted from: Mark Redekopp and David Kempe

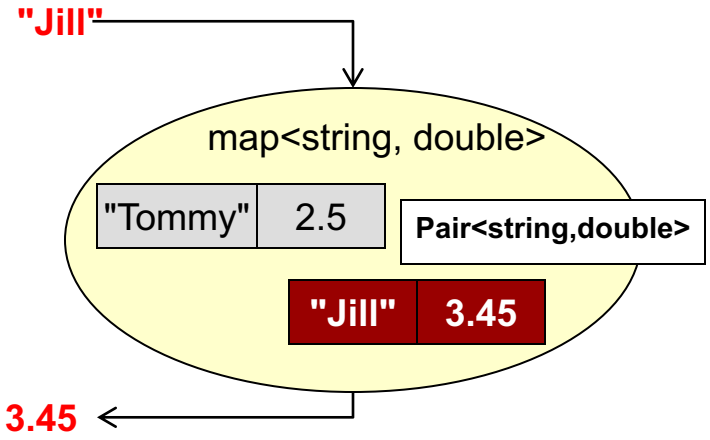
HASH TABLES

Dictionaries/Maps

- An array maps **integers** to values
  - Given i, array[i] returns the value in O(1)
- Dictionaries map **keys** to values
  - Given key, k, map[k] returns the associated value
  - Key can be anything provided…
    - It has a '<' operator defined for it (C++ map) or some other comparator functor
    - Most languages implementation of a dictionary implementation require something similar to operator< for key types



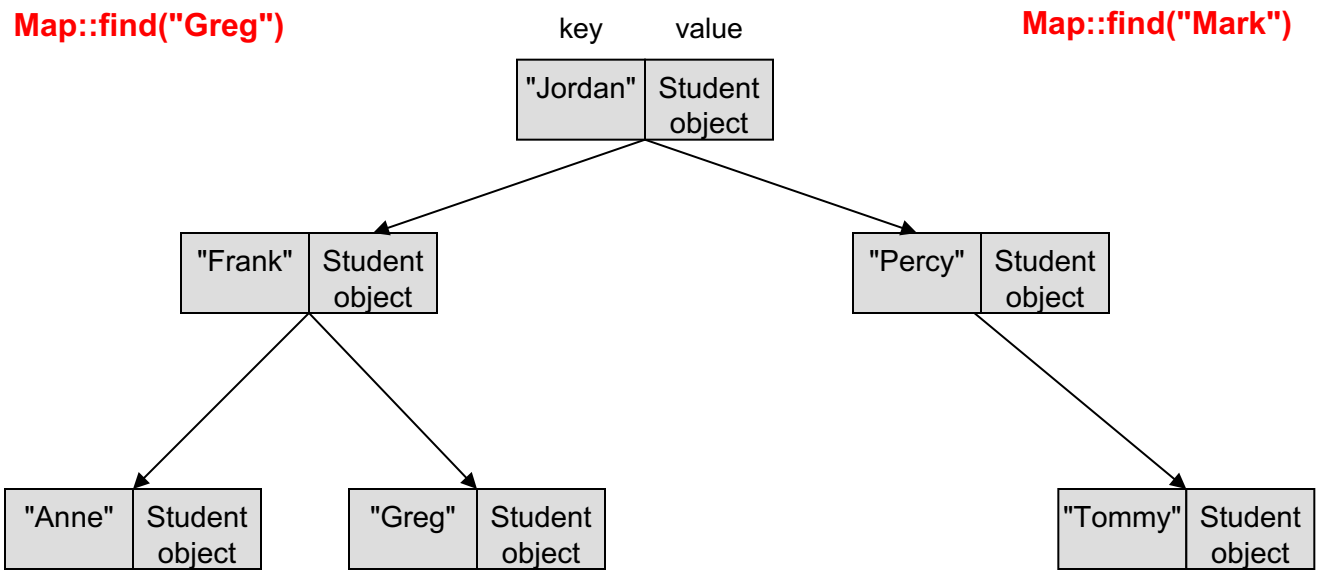
Arrays associate an integer with some arbitrary type as the value (i.e. the key is always an integer)



C++ maps allow any type to be the key

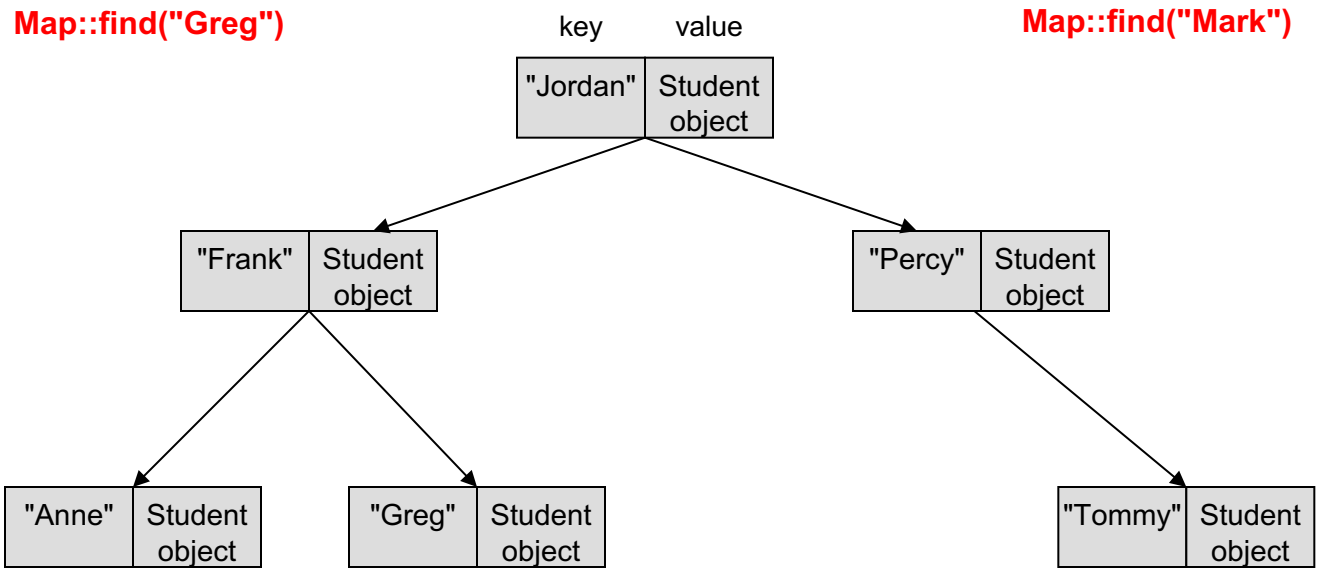
Dictionary Implementation

- A dictionary/map can be implemented with a balanced BST
  - Insert, Find, Remove = O(_____)



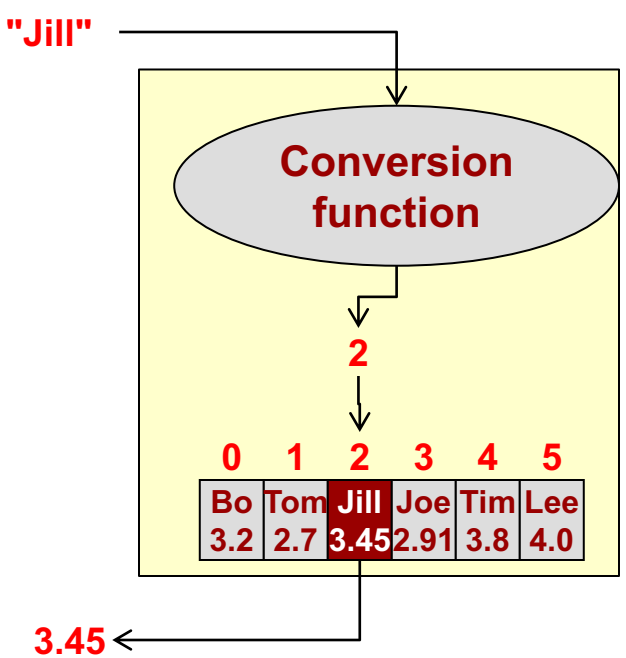
Dictionary Implementation

- A dictionary/map can be implemented with a balanced BST
  - Insert, Find, Remove = $O(\log_2 n)$
- Can we do better?
  - Hash tables (unordered maps) offer the promise of O(1) access time



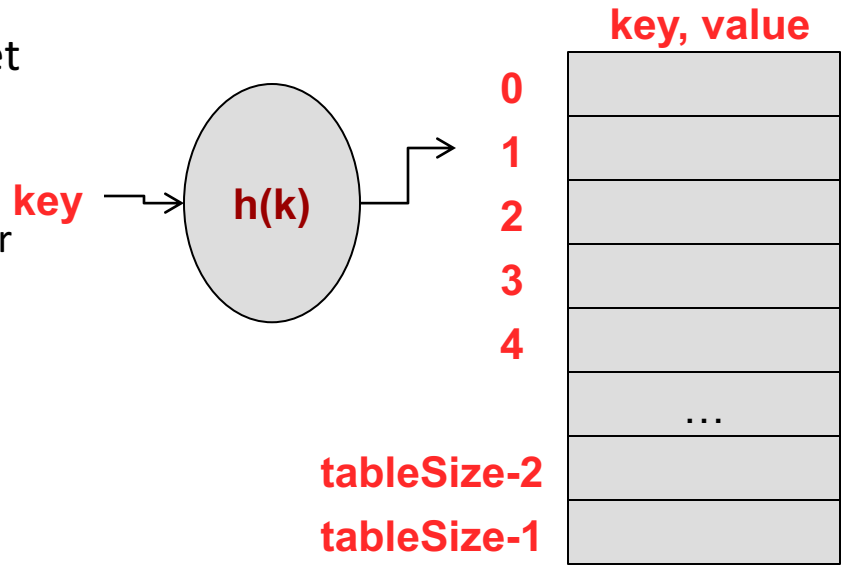
Hash Tables

- Can we use non-integer keys but still use an array?
- What if we just convert the non-integer key to an integer.
  - For now, make the unrealistic assumption that each unique key converts to a unique integer
- This is the idea behind a hash table
- The conversion function is known as a ***hash function, h(k)***
  - It should be fast/easy to compute (i.e. O(1))



Hash Tables

- A hash table is an array that stores key,value pairs
  - Usually smaller than the size of possible set of keys, |S|
    - USC ID's = 10^{10} options
    - Pick a hash table of some size much smaller (how many students do we have at any particular time)
- The table is coupled with a function, *h(k)*, that maps keys to an integer in the range [0..tableSize-1] (i.e. [0 to m-1])
- What are the considerations…
  - How big should the table be?
  - How to select a hash function?
  - What if two keys map to the same array location? (i.e. h(k1) == h(k2))
    - Known as a collision



Define
m = tableSize
n = # of used entries

Hash Functions First Look

- Define **N** = # of entries stored, **M** = Table/Array Size
- A hash function must be able to
  - convert the key data type to an integer
  - That integer must be in the range [0 to **M-1**]
    - Keeping h(k) in the range of the tableSize (**M**)
    - Fairly easy method: Use modulo arithmetic (i.e. h(k) % **M**)
- Usually converting key data type to an integer is a user-provided function
  - Akin to the operator<() needed to use a data type as a key for the C++ map
- Example: Strings
  - Use ASCII codes for each character and add them or group them
  - "hello" => 'h' = 104, 'e'=101, 'l' = 108, 'l' = 108, 'o' = 111 =
  - Example function: h("hello") = 104 + 101 + 108 + 108 + 111 = 532 **% M**

Hash Function Desirables

- A "perfect hash function" should map each given key to a unique location in the table
  - Perfect hash functions are not practically attainable
- A "good" hash function
  - Is easy and fast to compute
  - Scatters data uniformly throughout the hash table
    - P(h(k) = x) = 1/**M**

Table Size

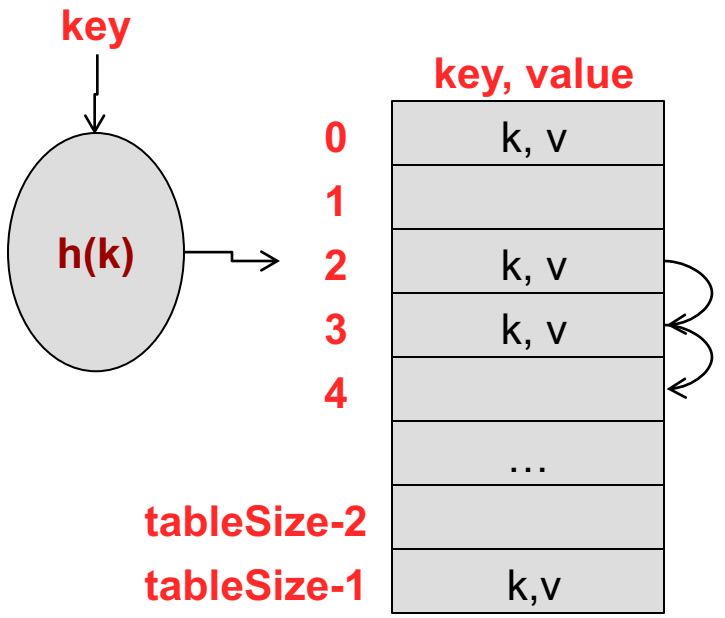
- Ideally…
  - Enough entries for all possible keys
  - Example: 3-letter airport codes: LAX, BUR, JFK would require how many table entries?
    - $26^3 = 17576$
    - Not all 3-letter codes correspond to airports
  - May be impractical as we will often only use a (small) subset of keys in a real application
- Realistically…
  - The table size should be bigger than the amount of expected entries
    - Don't pick a table size that is smaller than your expected number of entries
  - But anything smaller than the size of all possible keys admits the chance that two keys map to the same location in the table (a.k.a. ***COLLISION***)
  - You will see that tableSize should usually be a **prime number**

Resolving Collisions

- Example:
  - A hash table where keys are phone numbers: (XXX) YYY-ZZZZ
  - Obviously we can't have a table with 10^{10} entries
  - Should we define h(k) as the upper 3 or 4 digits: XXX or XXXY
    - Meaning a table of 1000 or 10,000 entries
  - Define h(k) as the lowest 4-digits of the phone number: ZZZZ
    - Meaning a table with 10,000 entries: 0000-9999
  - Now 213-740-4321 and 323-681-4321 both map to location 4321 in the table
- Collisions are hard to avoid so we have to find a way to deal with them
- Methods
  - Open addressing (probing)
    - Linear, quadratic, double-hashing
  - Buckets/Chaining (Closed Addressing)

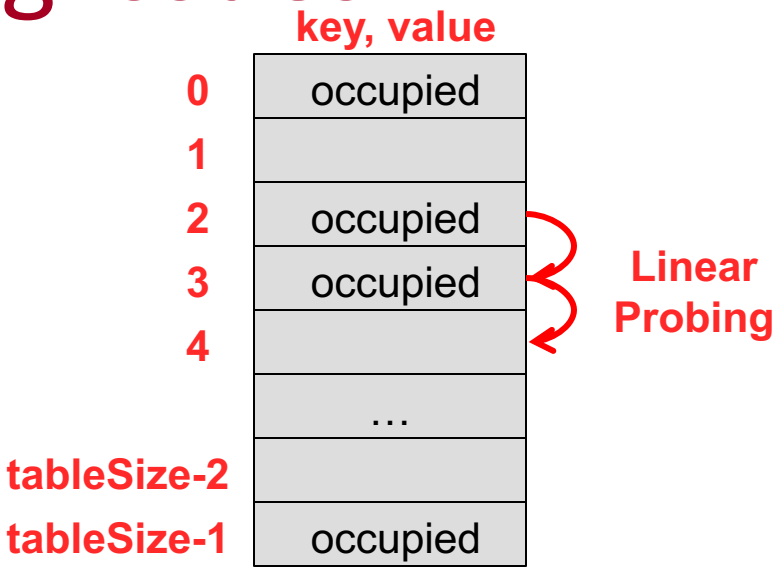
Open Addressing

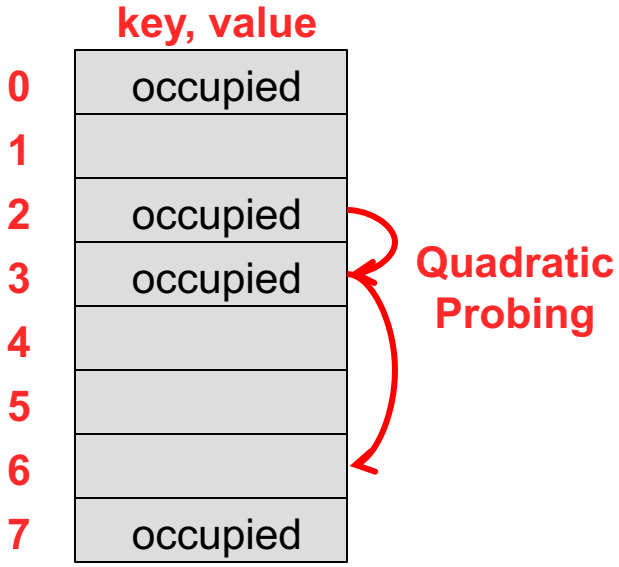
- Open addressing means an item with key, k, may not be located at h(k)
- Assume, location 2 is occupied with another item
- If a new item hashes to location 2, we need to find another location to store it
- Linear Probing
  - Just move on to location h(k)+1, h(k)+2, h(k)+3,…
- Quadratic Probing
  - Check location $h(k)+1^2$, $h(k)+2^2$, $h(k)+3^2$, …



Linear Probing Issues

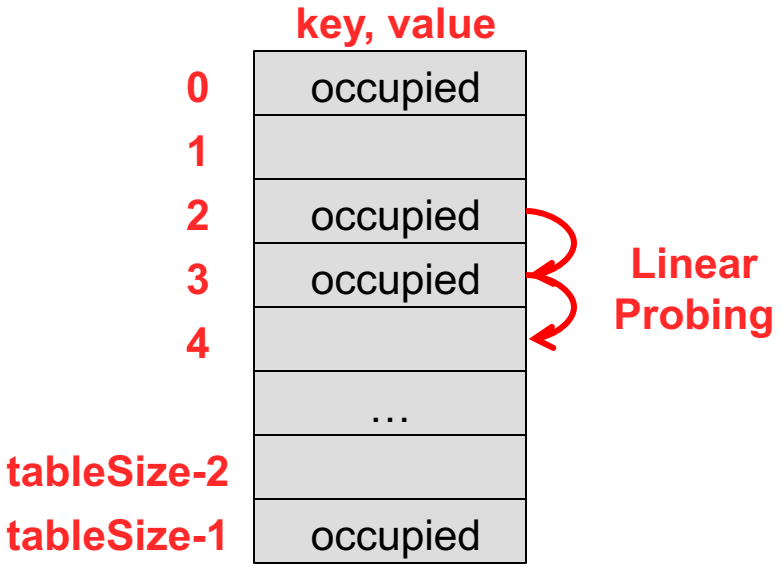
- If certain data patterns lead to many collisions, linear probing leads to clusters of occupied areas in the table called ***primary clustering***
- How would quadratic probing help fight primary clustering?
  - Quadratic probing tends to spread out data across the table by taking larger and larger steps until it finds an empty location

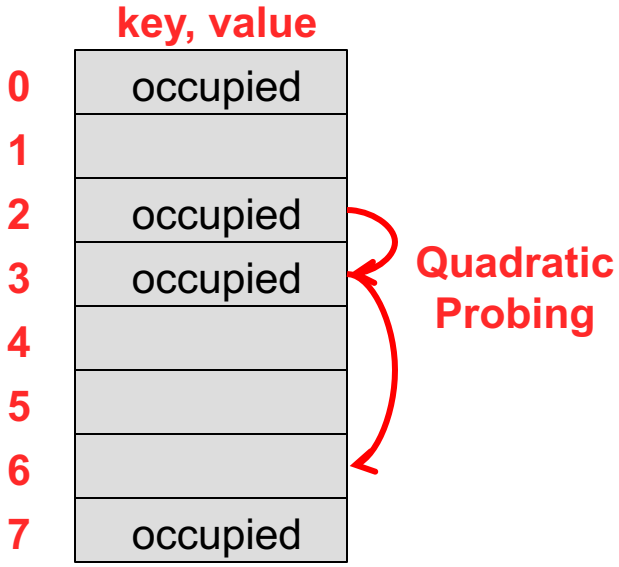




Find & Removal Considerations

- Given linear or quadratic clustering how would you find a given key, value pair
  - First hash it
  - If it is not at h(k), then move on to the next items in the linear or quadratic sequence of locations until
    - you find it or
    - an empty location or
    - search the whole table
- What if items get removed
  - Now the find algorithm might terminate too early
  - Mark a location as "removed"=unoccupied but part of a cluster





Practice

- Use the hash function h(k)=k%10 to find the contents of a hash table (m=10) after inserting keys 1, 11, 2, 21, 12, 31, 41 using linear probing

0	1	2	3	4	5	6	7	8	9
	1	11	2	21	12	31	41		

- Use the hash function h(k)=k%9 to find the contents of a hash table (m=9) after inserting keys 36, 27, 18, 9, 0 using quadratic probing

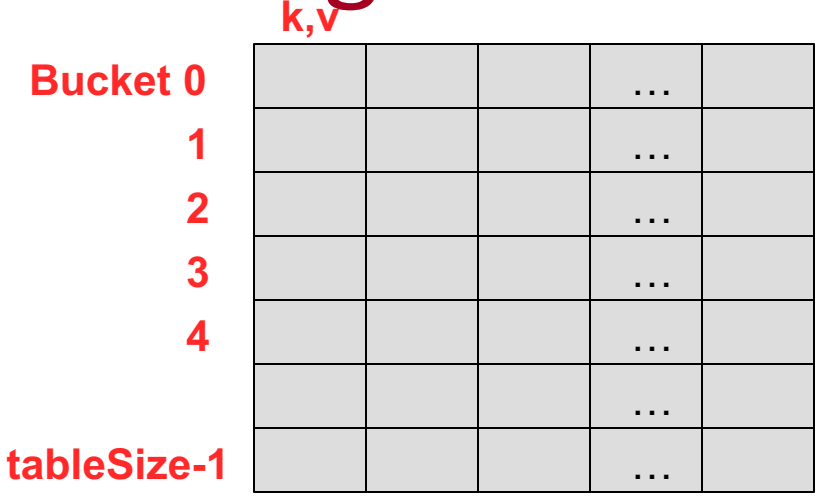
0	1	2	3	4	5	6	7	8
36	27	18	9	0				

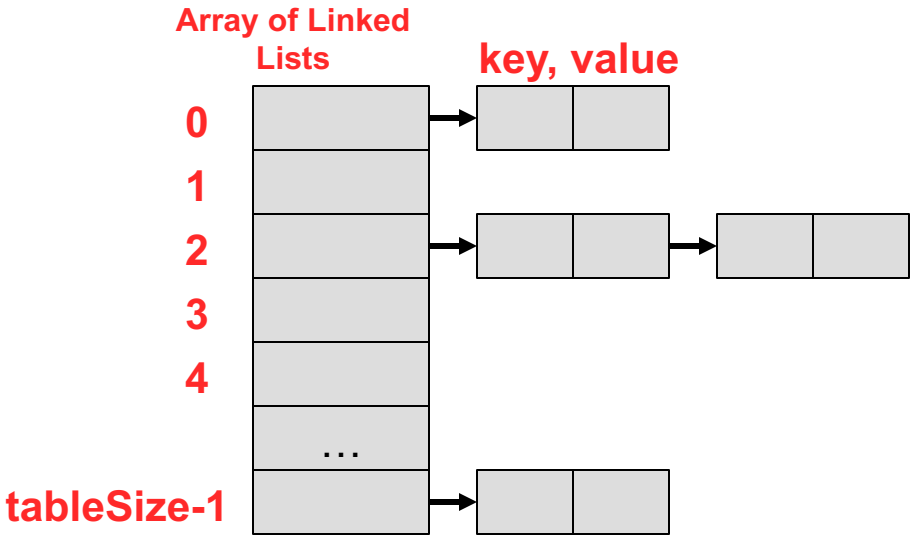
Double Hashing

- Define $h_1(k)$ to map keys to a table location
- But also define $h_2(k)$ to produce a linear probing step size
  - First look at $h_1(k)$
  - Then if it is occupied, look at $h_1(k) + h_2(k)$
  - Then if it is occupied, look at $h_1(k) + 2*h_2(k)$
  - Then if it is occupied, look at $h_1(k) + 3*h_2(k)$
- TableSize=13, h1(k) = k mod 13, and h2(k) = 5 – (k mod 5)
- What sequence would I probe if k = 31
  - h1(31) = 5, h2(31) = 5-(31 mod 5) = 4
  - 5, 9, 0, 4, 8, 12, 3, 7, 11, 2, 6, 10, 1

Buckets/Chaining

- Rather than searching for a free entry, make each entry in the table an ARRAY (bucket) or LINKED LIST (chain) of items/entries
- Buckets
  - How big should you make each array?
  - Too much wasted space
- Chaining
  - Each entry is a linked List





Hash Tables

- Suboperations
  - Compute h(k) should be O(1)
  - Array access of table[h(k)] = O(1)
- In a hash table, what is the expected efficiency of each operation
  - Find = O(1)
  - Insert = O(1)
  - Remove = O(1)

Hashing Efficiency

- Loading factor, α, defined as:
  - (N=number of items in the table) / M=tableSize => α = N / M
  - Really it is just the % of locations currently occupied
- For chaining, α, can be greater than 1
  - α = (number of items in the table) / tableSize
  - (number of items in the table) can be greater than tableSize
- What is the average length of a chain in the table? **α**
  - 10 total items in a hashTable of size 5 => expected chain = 2

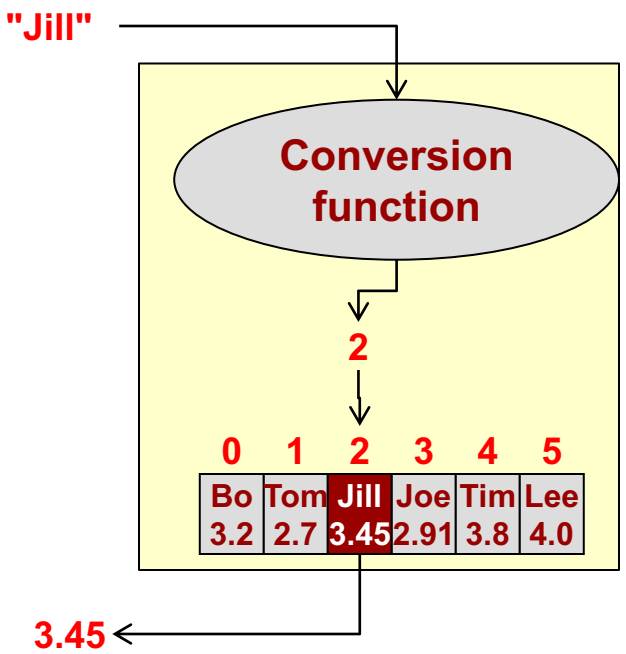
- # of operations to search
  - Unsuccessful search: 1 + α
  - Successful search: 1 + α/2

Rehashing for Open Addressing

- For open addressing/probing time also depends on α
- As α approaches 1, expected comparisons will get very large
  - Capped at the tableSize (i.e. O(n))
- Using a dynamic array (vector) we can simply allocate a bigger table
  - Don't just double it because we want the tableSize to be prime
- Can we just copy items over to the corresponding location in the new table?
  - No because the hash function usually depends on tableSize so we need to re-hash each entry to its correct location
  - h(k) = k % 13 != h'(k) = k %17 (e.g. k = 15)
- General guideline is to keep α < 1/2)

Unordered Maps

- A hash table implements a map ADT
  - Add(key,value)
  - Remove(key)
  - Lookup/Find(key)
    - Returns value
- Given a key, the hash function is interested in producing unique, evenly spread integers, NOT maintaining ordering of the keys
  - That is, just because k1 < k2, doesn't mean h(k1) < h(k2)
  - Thus the table holds values in arbitrary order unlike the BST
  - If you iterate through a hash table (and sometimes that is even a challenge), you likely won't see key,value pairs in order
- A hash table implements an **UNORDERED MAP**
- A Binary Search Tree implements an **ORDERED MAP**



C++11 Implementation

- C++11 added new container classes:
  - unordered_map
  - unordered_set
- Each uses a hash table for average complexity to insert , erase, and find in O(1)

HASH FUNCTIONS

Pigeon Hole Principle

- Recall for hash tables we let…
  - **n** = # of entries (i.e. keys)
  - **m** = size of the hash table
- If **n** > **m**, is every entry in the table used?
  - No. Some may be blank?
- Is it possible we haven't had a collision?
  - No. Some entries have hashed to the same location
  - Pigeon Hole Principle says given **n** items to be slotted into m holes and **n** > **m** there is at least one hole with more than 1 item
  - So if **n** > **m**, we know we've had a collision
- We can only avoid a collision when **n** < **m**

How Soon Would Collisions Occur

- Even if **n** < **m**, how soon would we expect collisions to occur?
- If we had an adversary…
  - Then maybe after the first two insertions
  - The adversary would choose 2 keys that mapped to the same place
- If we had a random assortment of keys…
- Birthday paradox
  - Given **n** random values chosen from a range of size **m**, we would expect a duplicate random value in O($\mathbf{m}^{1/2}$) trials
    - For actual birthdays where **m** = 365, we expect a duplicate within the first 23 trials

Hash Functions

- A "perfect hash function" should map each given key to a unique location in the table
  - Perfect hash functions are not practically attainable
- A "good" hash function
  - Is easy and fast to compute
  - Scatters data evenly throughout the hash table
  - Scatters **random** keys uniformly
    - **M**=3, keys=[0..4], h(k) = k % **M** does not spread data randomly
  - Scatters **clustered** keys uniformly
- Rules of thumb
  - The hash function should examine the entire search key, not just a few digits or a portion of the key
  - If modulo hashing is used, the base should be prime

Modulo Arithmetic

- Simple hash function is h(k) = k mod **m**
  - If our data is not already an integer, convert it to an integer first
- Recall **m** should be _____
  - PRIME!!!
- Say we didn't pick a prime number but some power of 10 (i.e. k mod 10^d) or power of 2 (i.e. 2^d)…then any clustering in the lower order digits would cause collisions
  - Suppose h(k) = k mod 100
  - Suppose we hash your birth years
  - We'd have a lot of collisions around _____
- Similarly in binary h(k) = k mod 2^d can easily be computed by taking the lower d-bits of the number
  - 17 dec. => 10001 bin. and thus 17 mod 2^2 = 01 bin.

Relatively Prime Numbers

- Two numbers are relatively prime if they do not share any factors other than 1
- If a and b (e.g. 9 and 7) are relatively prime, then their first common multiple is?
  - a*b
- If **m** (i.e. tableSize) is a prime number (not 2 or 5) what is the first common multiple of 10^d and **m**?
  - **m***10^d
  - For **m** = 11 and d=2, common multiples would be 1100, 2200, 3300, etc.

Why Prime Table Size

- Let's suppose we have clustered data when we chose **m**=10^d
  - Assume we have a set of keys, S = {k, k', k"…} (i.e. 99, 199, 299, 2099, etc.) that all have the same value mod 10^d and thus the original clustering (i.e. all mapped to same place when **m**=10^d
- Say we now switch and choose **m** to be a **prime** number (m=p)
- **What is the chance these numbers hash to the same location (i.e. still cluster) if we now use h(k) = (k mod m) [where m is prime]?**

Why Prime Table Size

- Notice if these numbers map to the same location when tableSize was $m=10^d$ then we know:
  - k mod 10^d = k' mod 10^d = r (i.e the same remainder for both)
- What can we say about the difference between any two keys that map to the same location? (e.g. k'-k)
  - They differ by some multiple of $m=10^d$ (i.e. the table size)
    - 2099-99 = 2000 = $20*10^2$, 2099-199 = 1900 = $19*10^d$, 199-99 = 100 = $1*10^2$
  - To repeat: if k and k' collide, k'-k will yield some multiple of 10^d
- Proof: Recall two numbers, (k and k') will hash to the same location if
  - k mod m = k' mod m = r (i.e the same remainder for both)
  - If that's true then we can write:
    k = qm + r and k' = q'm + r for some q and q' where $q \neq q'$
  - And thus k'-k = (q'-q)*m….the difference is some multiple of m
  - So for all these numbers k'-k yielded some multiple of 10^d
    - Put another way the stepsize = 10^d

Modulo Hashing

- So to map to the same place in the $m=10^d$ hash table, k-k' would have to be a multiple of 10^d
- BUT…if k and k' were to map to the same place using our new tableSize (m=some prime number, p) then k-k' (which we know is a power of 10^d because they collided when $m = 10^d$) would ALSO have to be divisible by p
  - So k-k' would have to be a multiple of 10^d and p
  - Recall what would the first common multiple of p and 10^d be?

- So for k and k' to map to the same place k-k' would have to be some multiple $p*10^d$
  - i.e. $1*p*10^d$, $2*p*10^d$, $3*p*10^d$, …
  - For p = 11 and d=2 => k-k' would have to be 1100, 2200, 3300, etc.
    - Ex. k = 1199 and k'=99 would map to the same place mod 11 and mod 10^2
    - Ex. k = 2299 and k'=99 would also map to the same place in both tables

Here's the Point

- Here's the point…
  - For the values that used to ALL map to the same place like 99, 199, 299, 399…
  - Now, only **every m-th** one maps to the same place (99, 1199, 2299, etc.)
  - This means the chance of clustered data mapping to the same location when m is prime is 1/m
  - In fact 99, 199, 299, 399, etc. map to different locations mod 11
- **So by using a prime tableSize (m) and modulo hashing even clustered data in some other base is spread across the range of the table**
  - **Recall a good hashing function scatters even clustered data uniformly**
  - **Each k has a probability 1/m of hashing to a location**

Another Alternative

- We just said a "good" hashing function provides a uniform probability distribution of hashing to any location
- So given a key, k, can you think of another hash function, h(k) such that k has a uniform probability of ending up in any location?
- How about h(k) = rand() % m
- Pros:
  - Fast
  - Even clustered keys get spread randomly over the hash table
- Cons:
  - How do you do a look up?
  - h(k) is not dependent on k…it is just based on the next random number

http://www.cs.cmu.edu/~avrim/451f11/lectures/lect1004.pdf

Universal Hash Functions

- One alternative to the above is not to make the hash-function random but to make the **selection** of the hash function random
- Anytime we fix on one hash function, an adversary can easily create keys that map to the same place
  - I.e. they could study the function and try to come up with a "bad" sequence
- Instead, suppose…
  - We have a family of "good" hash functions (h1, h2, h3…) where each hash function is independent of the others and has the probability of mapping two keys to the same place = 1/m
  - Certainly an adversary could design a "bad" sequence of keys for each of these "good" hash functions.
  - But now, when we want to create a hash table we simply randomly select which hash function to use at run-time
  - The adversary wouldn't know which one we picked and thus couldn't know which of his bad sequences to feed us…
  - We've essentially made the odds of a "bad" sequence improbable
  - And wasn't that the intention of a hash function anyway?
  - These families of hash functions are known as **universal hash functions**

Taking a Step Back

- How can any deterministic hash function appear to spread items randomly over the hash table?
- The pigeon-hole principle says that given N items for M slots where N>M, at least one hole will have N/M items
  - 1000 items for 100 slots…one slot is guaranteed to have 10 or more
- In reality the POSSIBLE values of N >> M
  - E.g. Possible universe of USC ID's (10^{10} options) mapping to a table on the order of 100,000
  - At least $10^{10}/10^5$ could map to the same place no matter how "good" your hash function…it's deterministic
  - What if an adversary fed those in to us…

Inverse Hash Function

- H(k) = c = k mod 11
  - What would be an adversarial sequence of keys to make my hash table perform poorly?
- It's easy to compute the inverse, $h^{-1}(c) \Rightarrow k$
  - Write an expression to enumerate an adversarial sequence?
  - 11*i + c for i=0,1,2,3,…
- We want hash function, h(k), where an inverse function, $h^{-1}(c)$ is **hard** to compute
  - Said differently, we want a function where given a location, c, in the table it would be hard to find a key that maps to that location
- We call these functions **one-way hash functions**
- **Key insight: What's hard to accomplish when you actually try is even harder to accomplish when you do not try**

One-Way Hash Functions

- **Recall: What's hard to accomplish when you actually try is even harder to accomplish when you do not try**

- So if we find a hash function where it would be hard to find a key that maps to a given location, i, when we are trying to be an adversary…
- …then under normal circumstances we would not expect to accidentally (or just in nature) produce a sequence of keys that leads to a lot of collisions

- Main Point: It's obviously hard to predict randomness, but computers aren't random. However, if we can find a one-way hash function, then even though our adversary knows we're not being random, he'll still have a hard time

Cryptographic Hash Functions

- Hash functions can be used for purposes other than hash tables
- We can use a hash function to produce a "digest" (signature, fingerprint, checksum) of a longer message
  - It acts as a unique "signature" of the original content
- The hash code can be used for purposes of authentication and validation
  - Send a message, m, and h(m) over a network.
  - The receiver gets the message, m', and computes h(m') which should match the value of h(m) that was attached
  - This ensures it wasn't corrupted accidentally or changed on purpose
- We no longer need h(m) to be in the range of tableSize since we don't have a table anymore
  - The hash code is all we care about now
  - We can make the hash code much longer (64-bits => 16E+18 options, 128-bits => 256E+36 options) so that chances of collisions are hopefully miniscule (more chance of a hard drive error than a collision)

Another Example: Passwords

- Should a company just store passwords plain text?
  - No
- We could encrypt the passwords but here's an alternative
- Just don't store the passwords!
- Instead, store the hash codes of the passwords.
  - What's the implication?
  - Some alternative password might just hash to the same location but that probability is even smaller than someone trying to hack your system of encrypted passwords
    - Remember the idea that if its hard to do when you try, the chance that it naturally happens is likely smaller
  - When someone logs in just hash the password they enter and see if it matches the hashcode.
- If someone gets into your system and gets the hash codes, does that benefit them?
  - No!

Cryptographic Hash Functions

- A cryptographic hash function should have the qualities
- Given d, it is hard to find an input, M, such that h(M) = d
  - (i.e. One-way: given a hash code, hard to find an input that generates it)
- Collision Resistant
  - It should be hard to generate M1 and M2 such that h(M1) = h(M2)
  - Remember what the birthday paradox says?
    - An n-bit code has 2^n options…if we generate random numbers when would we expect a duplicate?
- Given an M1 and h(M1) = d, it should be hard to find an M2 such that h(M2) = d
  - Given one input and its hash code it should be hard to tweak M1 to make an M2 that yields an identical code
- MD5 and SHA2 are examples of these kind of functions

http://people.csail.mit.edu/shaih/pubs/Cryptographic-Hash-Functions.ppt

Avalanche Effect

- A desirable property for a cryptographic hash function is that it exhibit the "avalanche effect"
- Avalanche effect says…
  - **Formally**: A change of 1-bit in the input should produce on average half of the output-bits to change (i.e. probability of an output bit change due to that input change should be ½)
  - **Informally**: A small change in input should yield a big change in output
- Along that same line, think about which bit-wise logic operation would be best suited for encryption
  - Ex. If I told you which logic operation I used and you saw the output, would it help you decipher the input that generated that?
- XOR's are often used in cryptographic hash function along with bit shifting and rotations

X1	X2	AND
0	0	0
0	1	0
1	0	0
1	1	1

X1	X2	OR
0	0	0
0	1	1
1	0	1
1	1	1

X1	X2	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Example

- Encrypt: "ONE MORE HW" to produce a 5-bit hash by XOR-ing the codes

Char.	Bin.	Char.	Bin.	Char.	Bin.	Char.	Bin.	Char.	Bin.
A	00000	F	00101	K	01010	P	01111	U	10100
B	00001	G	00110	L	01011	Q	10000	V	10101
C	00010	H	00111	M	01100	R	10001	W	10110
D	00011	I	01000	N	01101	S	10010	X	10111
E	00100	J	01001	O	01110	T	10011	Y	11000
Sp	11010							Z	11001

Summary

- Hash tables are LARGE arrays with a function that attempts to compute an index from the key
- In the general case, **chaining** is the best collision resolution approach
- The functions should spread the possible keys evenly over the table