Mining Social Network Graphs

Analysis of Large Graphs: Community Detection

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With slide contributions from
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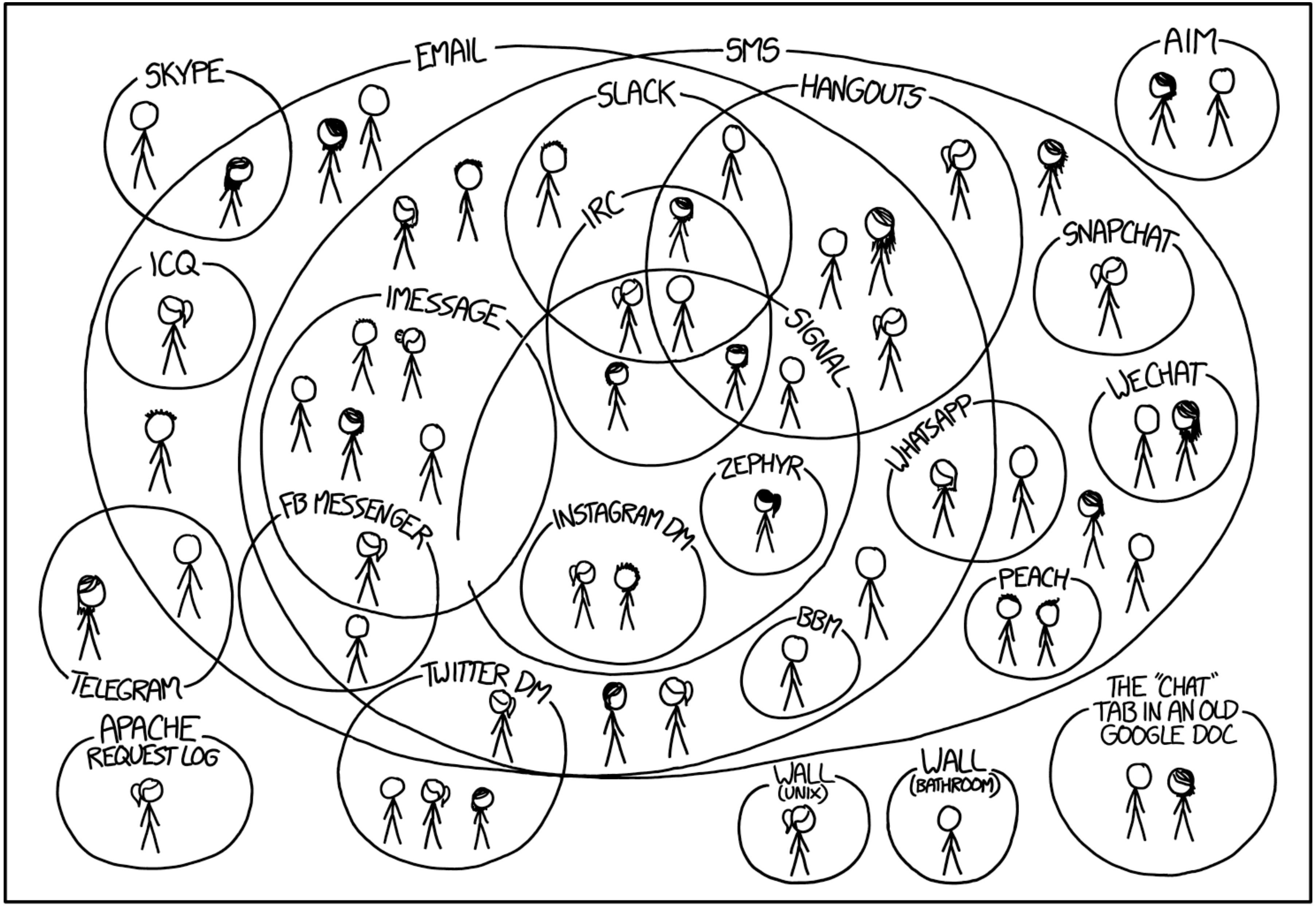


I HAVE A HARD TIME KEEPING TRACK OF WHICH CONTACTS USE WHICH CHAT SYSTEMS.

https://xkcd.com/1810/

Social Networks: Examples

- Examples

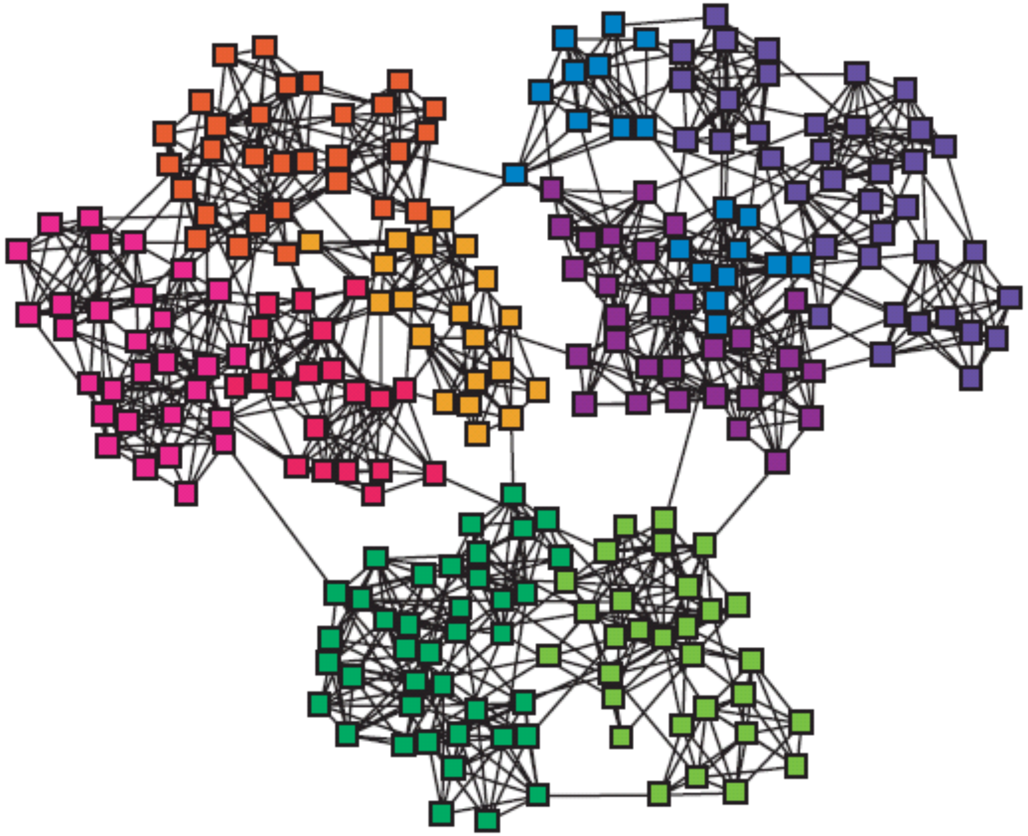
```
docker run -p 8888:8888 jupyter/scipy-notebook:2c80cf3537ca
```

- Use files:

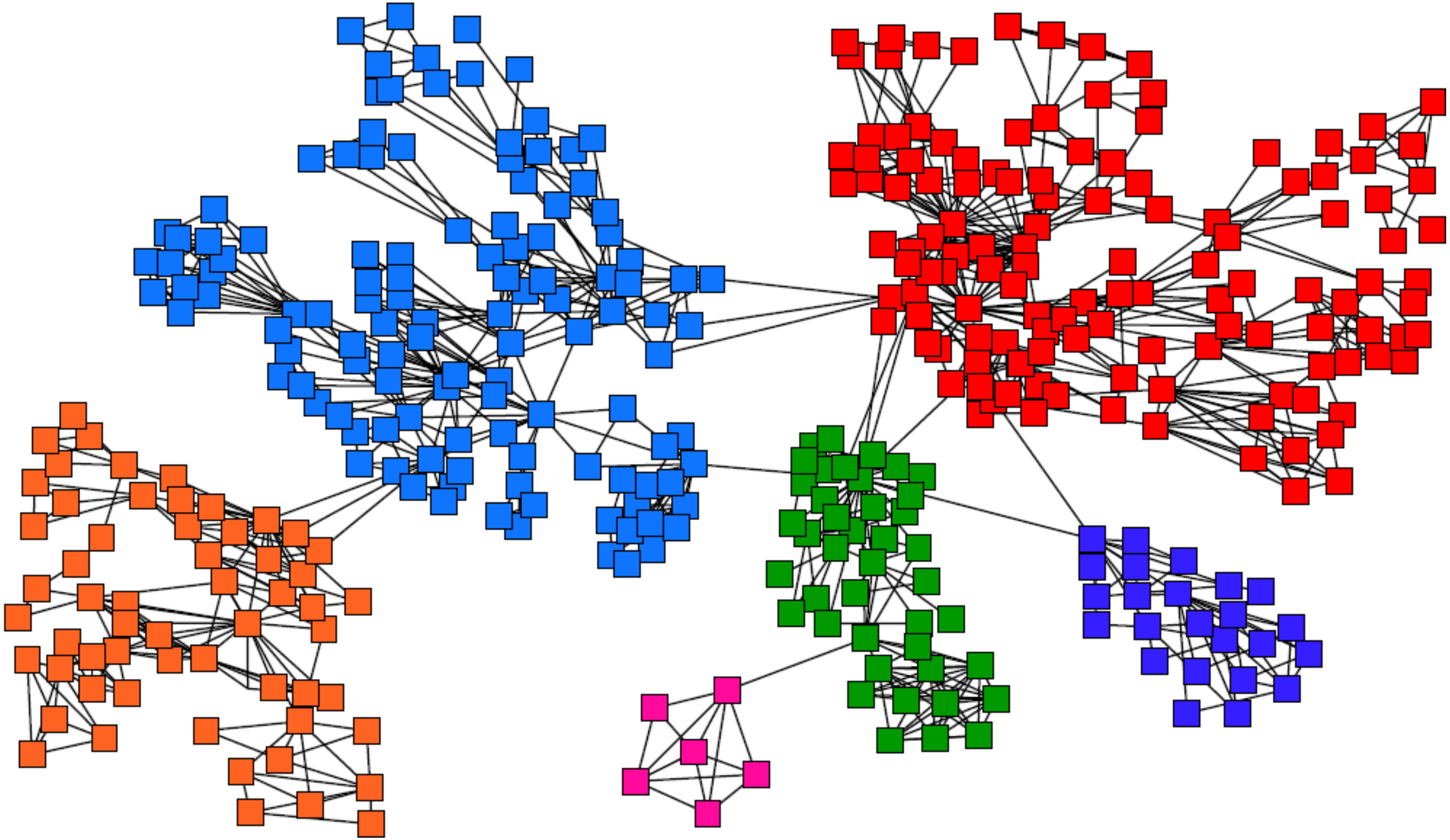
```
Social-Networks-Networkx.ipynb
modularity_maximization.zip
```

Networks & Communities

- **We often think of networks being organized into modules, cluster, communities:**

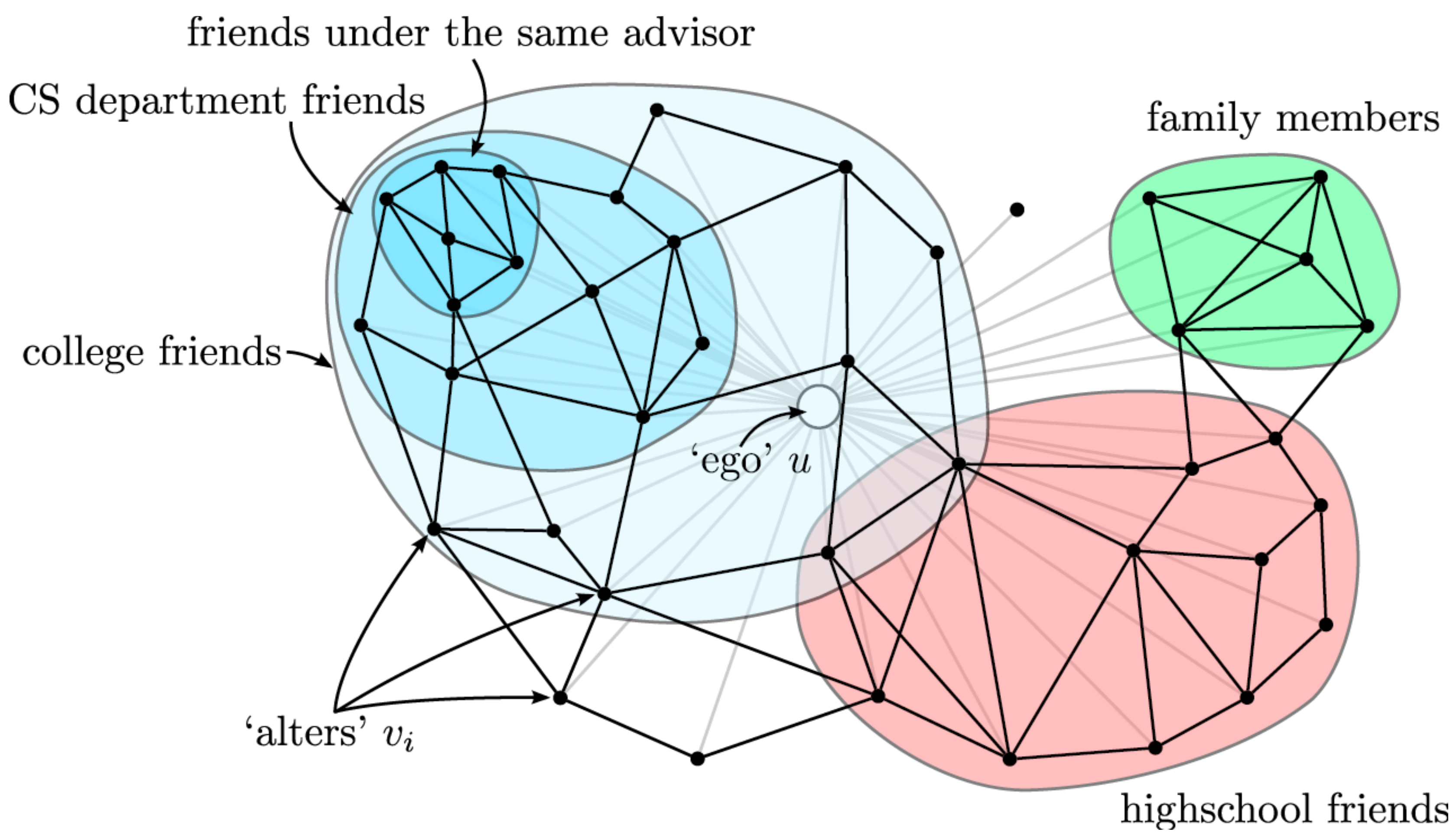


Goal: Find Densely Linked Clusters



Twitter & Facebook

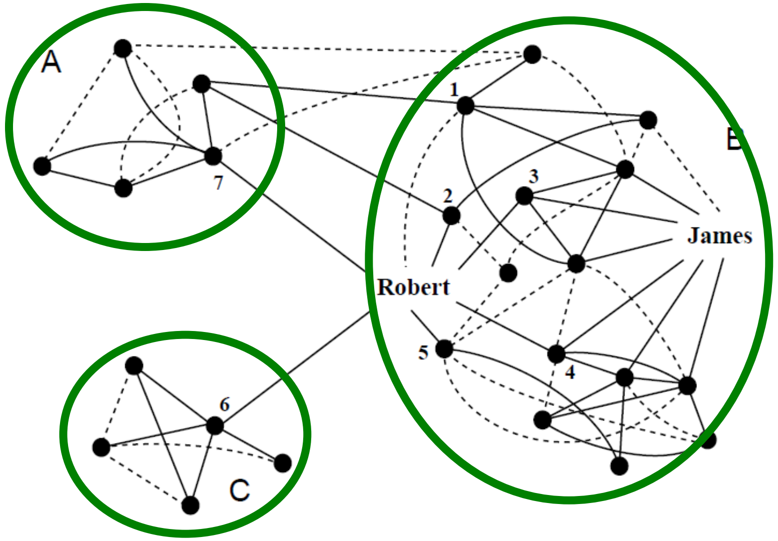
- **Discovering social circles, circles of trust:**

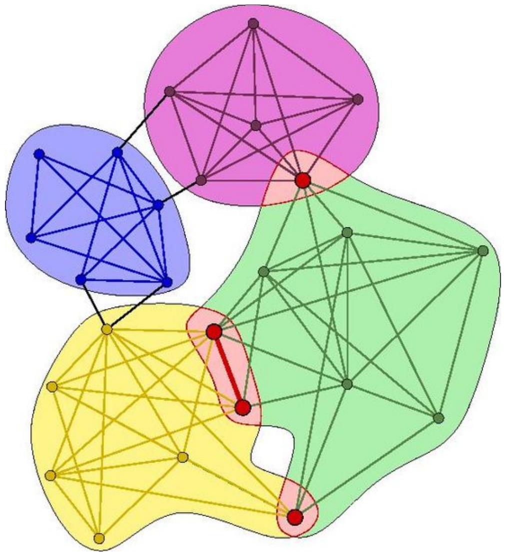


[McAuley, Leskovec: Discovering social circles in ego networks, 2012]

Community Detection (Graph Basics)

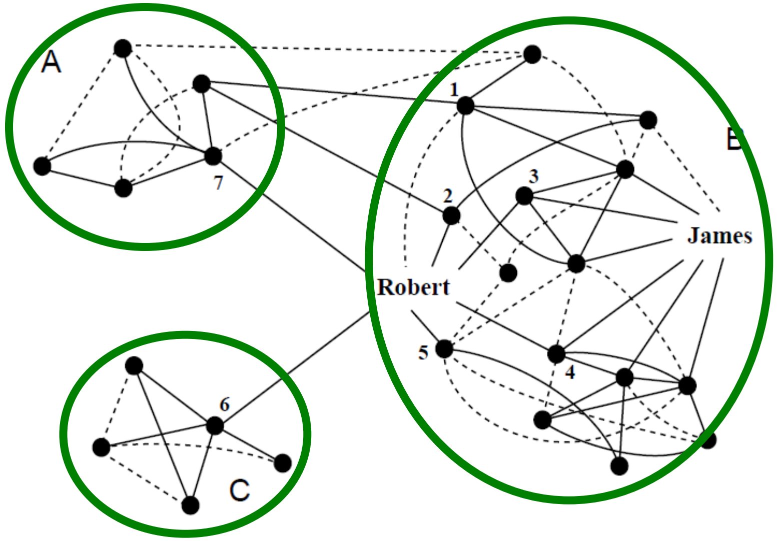
How to find communities?

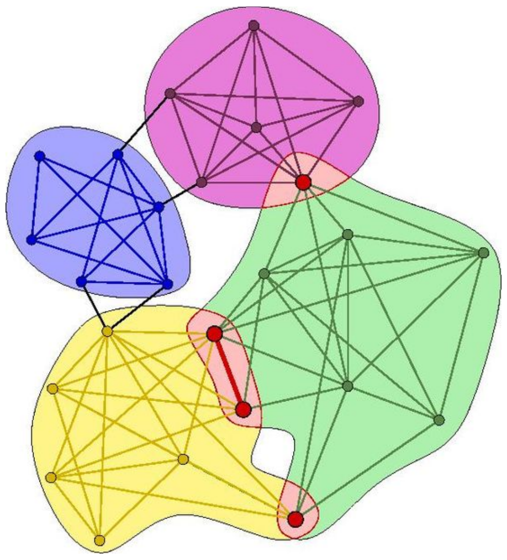




Community Detection (Algorithms and Methods)

How to find communities?

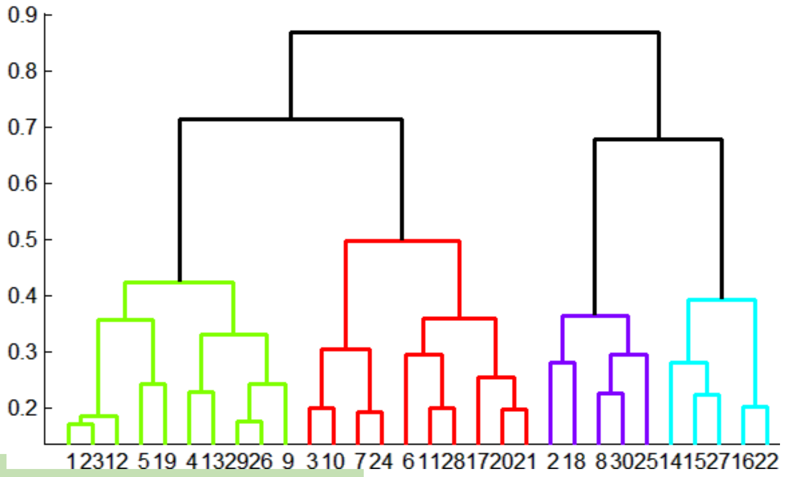




We will work with **undirected** (unweighted) networks

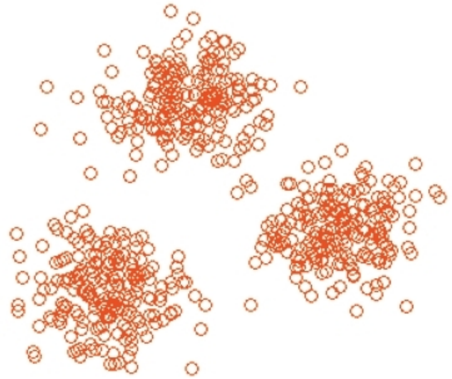
Recall: Methods of Clustering

- **Hierarchical:**
  - **Agglomerative** (bottom up):
    - Initially, each point is a cluster
    - Repeatedly combine the two "nearest" clusters into one
    - **Used a distance metric**



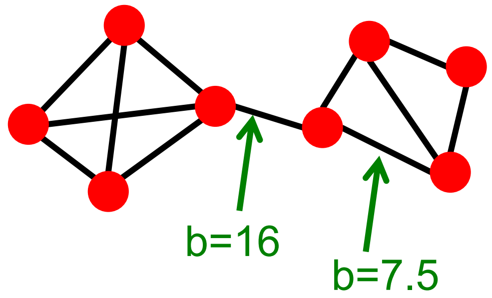
Today:
  - **Divisive** (top down):
    - Start with one cluster and recursively split it

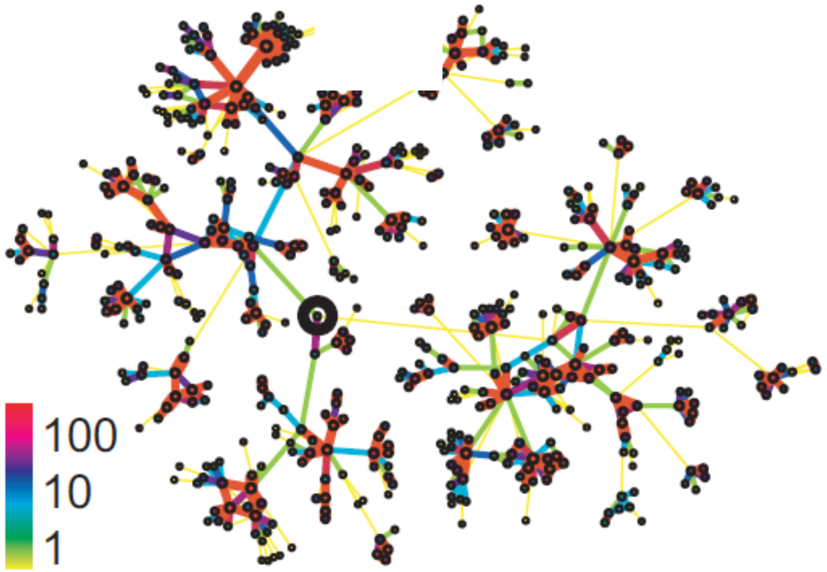
- **Point assignment:**
  - Maintain a set of clusters
  - Points belong to "nearest" cluster
  - **Used a distance metric**



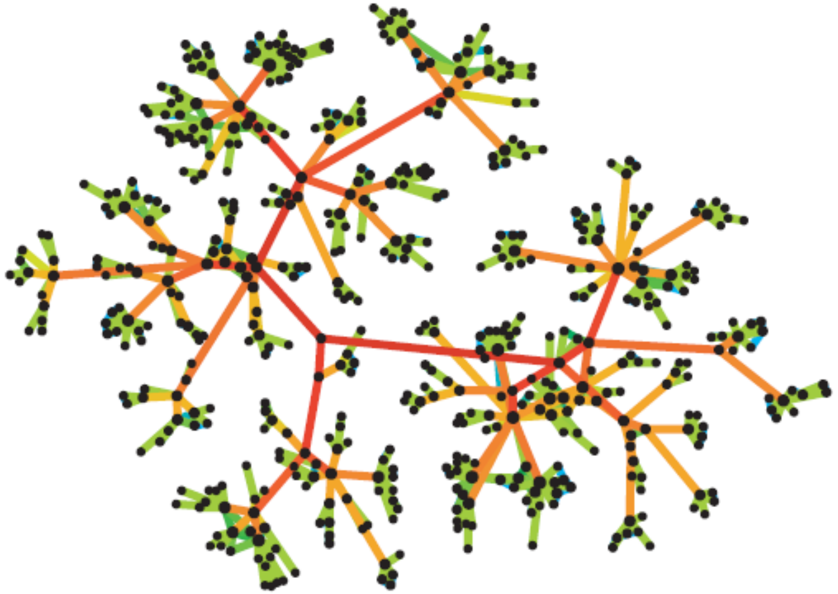
Betweenness Concept

- **Edge betweenness:** **Number of shortest paths passing over the edge**
- **Intuition:**





Edge strengths (call volume) in a real network

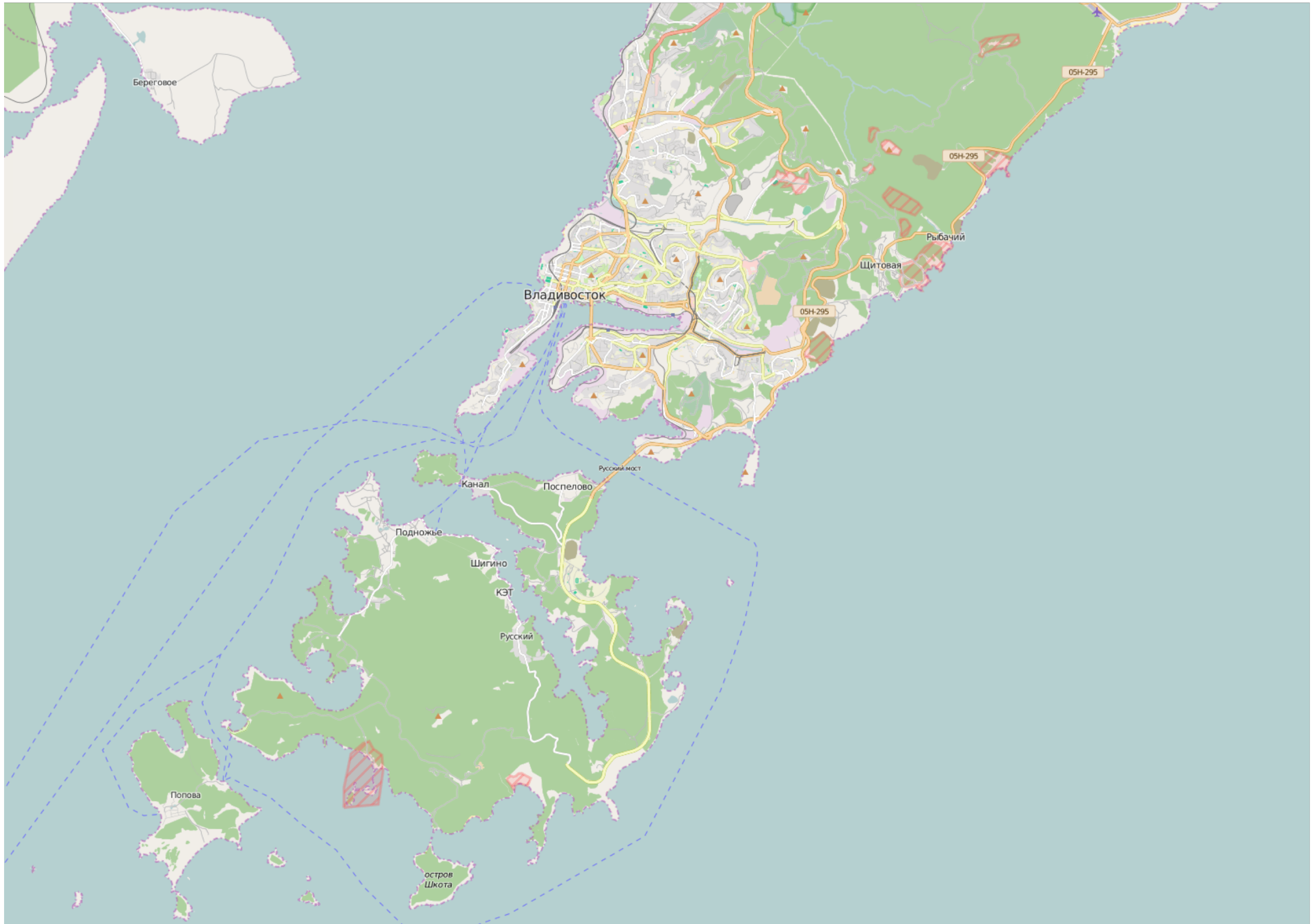


Edge betweenness in a real network

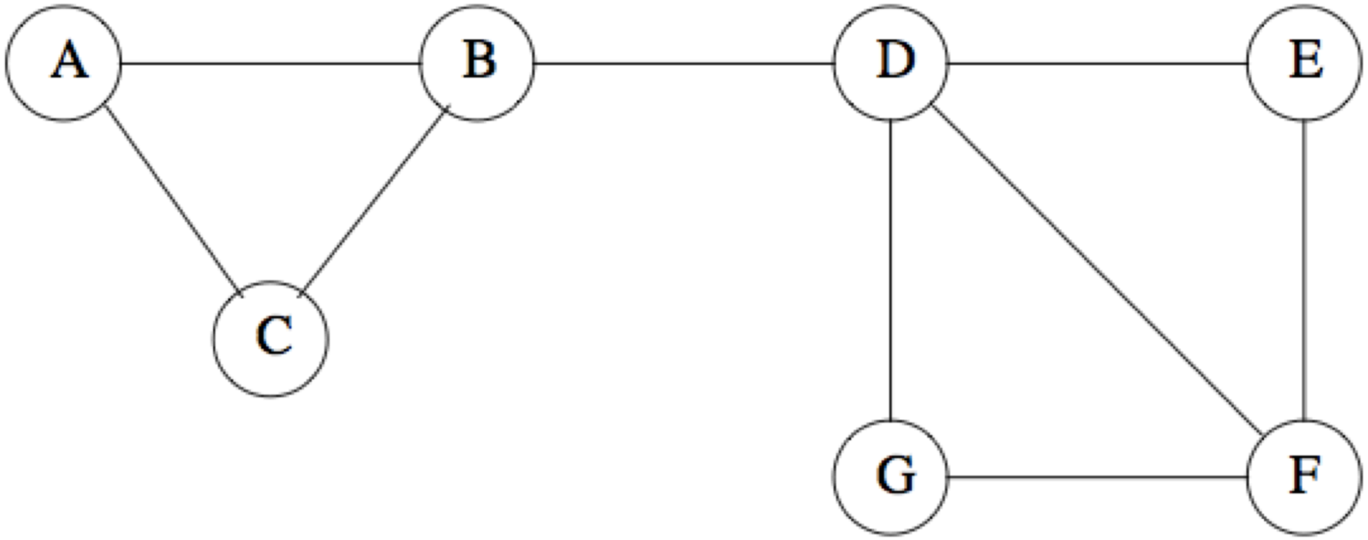
Betweenness Concept (Cont'd)

- Find edges in a social network graph that are least likely to be inside a community
- Betweenness of edge (a, b):
  - number of pairs of nodes x and y -> x, y $\in C$
  - **edge (a,b) lies on the shortest path between x and y**
- If there are several shortest paths between x and y, edge (a,b) is credited with the fraction of those shortest paths that include edge (a,b)
- **A high score is bad:** suggests that edge (a,b) runs between two different communities
  - **a and b are in different communities**

The Russian Bridge



Betweenness Example



- Expect that edge (B,D) has highest betweenness
- (B,D) is on every shortest path from {A,B,C} to {D,E,F,G}
- Betweenness of (B,D) = 3x4 = 12
- (D,F) is on every shortest path from {A,B,C,D} to {F}
- Betweenness of (D,F) = 4x1 = 4
- **Natural communities: {A,B,C} and {D,E,F,G}**

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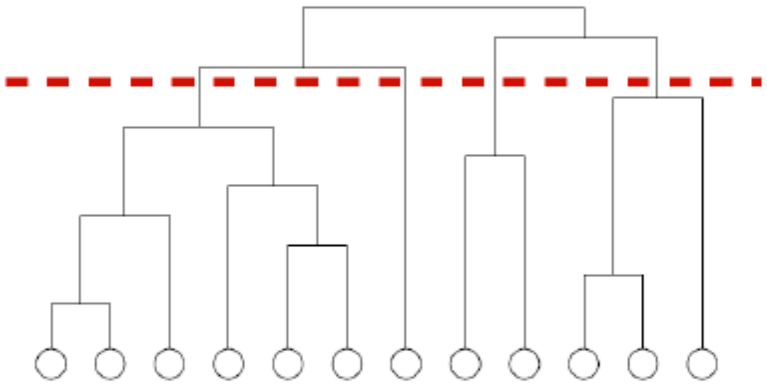
We need to resolve 2 questions

1. **How to compute betweenness?**
2. **How to select the number of clusters?**



The Girvan-Newman Algorithm

- Want to **discover communities using divisive hierarchical clustering**
  - Start with one cluster (the social network) and recursively split it
- **Will do this** based on the notion of edge **betweenness**:

  Number of shortest paths passing through the edge
- **Girvan-Newman Algorithm:**
  - Visits each node X once
  - Computes the number of shortest paths from X to each of the other nodes that go through each of the edges
- **Repeat:**
  - Calculate betweenness of edges
    1. Thresholding to remove high betweeness edges, or
    2. Remove edges with highest betweenness: **between** communities
- **Connected components are communities**
- Gives a hierarchical decomposition of the network

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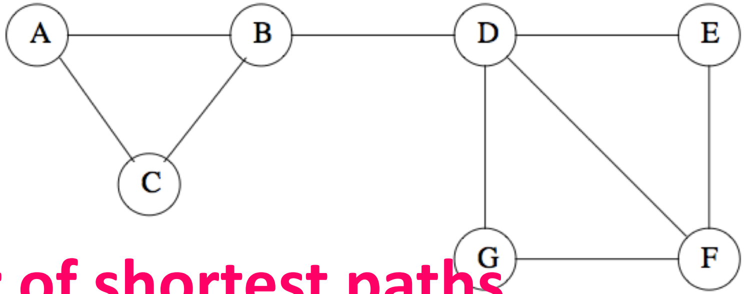
Girvan-Newman Algorithm (1)

- Visit each node X once and **compute the number of shortest paths from X to each of the other nodes that go through each of the edges**

- **1) Perform a breadth-first search (BFS) of the graph, starting at node X**
  - The level of each node in BFS is length of the shortest path from X to that node
  - So edges that go between nodes on the same level can never be part of a shortest path from X
  - **Edges between levels are called DAG edges** (DAG = Directed Acyclic Graph)
  - **Each DAG edge is part of at least one shortest path from root X**

Girvan-Newman Algorithm (2)



- **2) Label each node by the number of shortest paths that reach it from the root node**
  - **Example: BFS starting from node E, labels assigned**

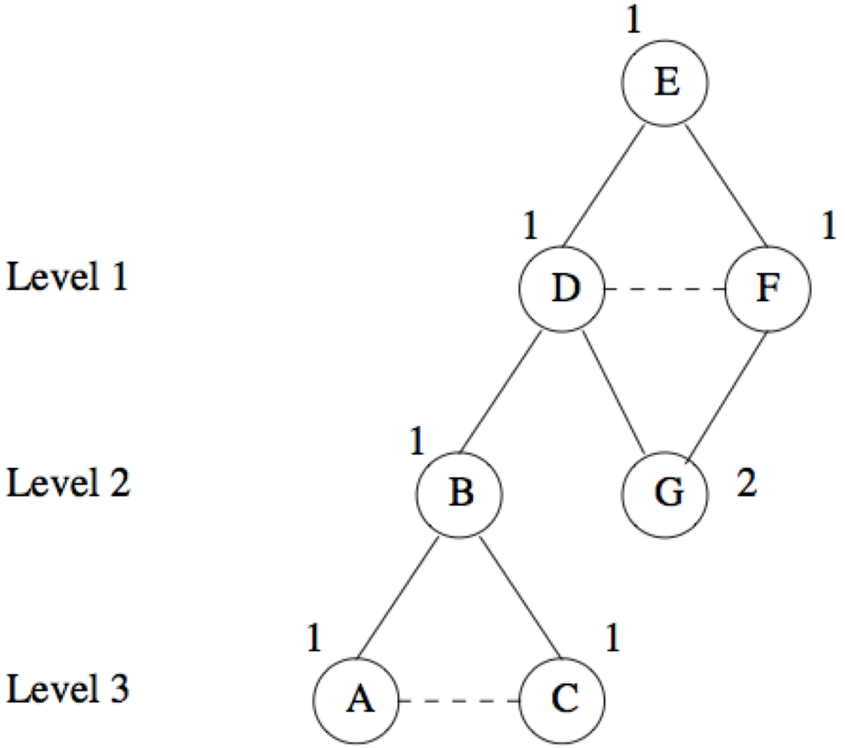


Figure 10.4: Step 1 of the Girvan-Newman Algorithm

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Girvan-Newman Algorithm (3)

- **3) Calculate for each edge *e*, the sum over all nodes Y (of the fraction) of the shortest paths from the root X to Y that go through edge *e***
  - **Compute this sum for nodes and edges, starting from the bottom of the graph**
  - **Each node** other than the root node **is given a credit of 1**
  - **Each leaf node in the DAG gets a credit of 1**
  - Each **node** that is not a leaf gets credit = **1 + sum of credits of the DAG edges from that node to level below**
  - A DAG **edge** e entering node Z (from the level above) is **given a share of the credit of Z proportional to the fraction of shortest paths from the root to Z that go through e**

Girvan-Newman Algorithm (4)

- Assign node and edge values starting from bottom

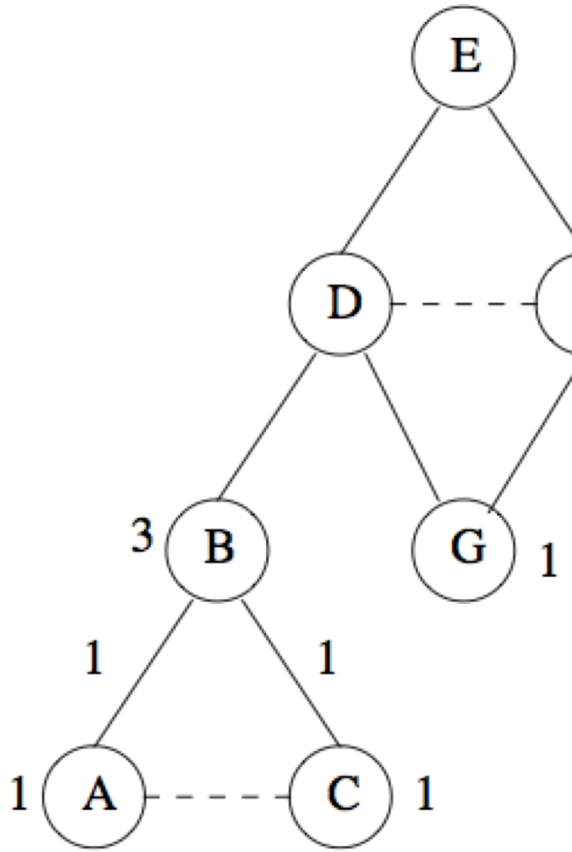
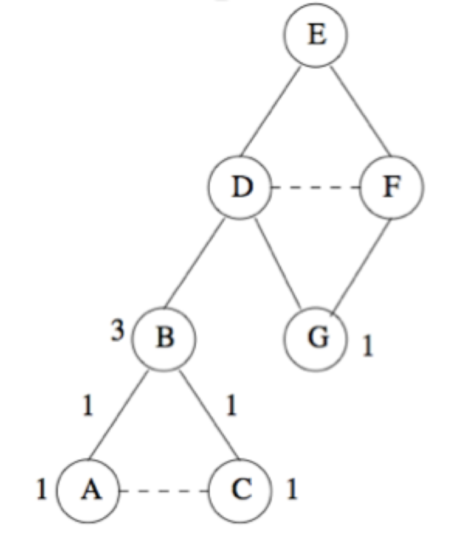


Figure 10.5: Final step of the Girvan-Newman Algorithm – levels 3 and 2

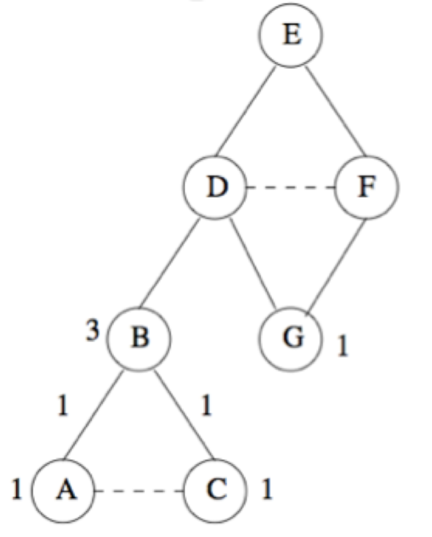
Girvan-Newman Algorithm (5)



Assigning credits:

- A and C are **leaves**: get credit = 1
- Each of these nodes **has only one parent**, so their credit=1 is **given to edges** (B,A) and (B,C)
- At level 2, G is a **leaf**: gets credit = 1
- B gets credit **1 + credit of DAG edges entering from below** = 1 + 1 +1 = 3
- B has only **one parent**, so edge (D,B) **gets entire credit of node** B = 3
- **Node G has 2 parents (D and F): how do we divide credit of G between the edges?**

Girvan-Newman Algorithm (6)



- In this case, **both D and F have just one path from E to each of those nodes**
  - So, **give half credit of node** G **to each of those edges**
  - Credit = 1/(1 + 1) = 0.5
- **In general, how we distribute credit of a node to its edges depends on number of shortest paths**
  - **Say there were 5 shortest paths to D and only 3 to F**
  - **Then credit of edge (D,G) = 5/8 and credit of edge (F,G) = 3/8**
- Node D gets credit = **1 + credits of edges below it** =
  1 + 3 + 0.5 = 4.5
- Node F gets credit = 1 + 0.5 = 1.5
- D has **only one parent**, so Edge (E,D) gets credit = 4.5 from D
- Likewise for F: Edge (E,F) gets credit = 1.5 from F

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Girvan-Newman Algorithm (7): Completion of Credit Calculation starting at node E

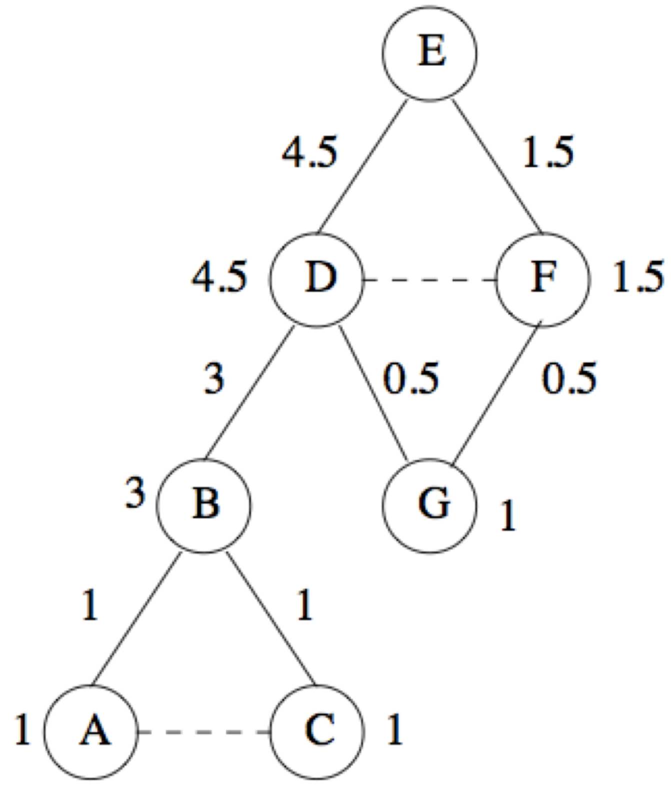


Figure 10.6: Final step of the Girvan-Newman Algorithm – completing the credit calculation

Girvan-Newman Algorithm (8): Overall Betweenness Calculation

- To complete betweenness calculation, must:
  - **Repeat this for every node as root**
  - **Sum the contributions** on each edge
  - **Divide by 2** to get true betweenness
    - since every shortest path will be counted twice, once for each of its endpoints

Using Betweenness to Find Communities: Clustering

- **Betweenness scores for edges of a graph behave something like a distance metric**
  - Not a true distance metric
- **Could cluster by taking edges in increasing order of betweenness and adding to graph one at a time**
  - At each step, connected components of graph form clusters
- **Girvan-Newman: Start with the graph and all its edges and remove edges with highest betweenness**
  - **Continue until graph has broken into suitable number of connected components**
  - **Divisive hierarchical clustering** (top down)
    - Start with one cluster (the social network) and recursively split it

Using Betweenness to Find Communities (2)

- (B,D) has highest betweenness (12)
- Removing edge would give natural communities we identified earlier: {A,B,C} and {D,E,F,G}

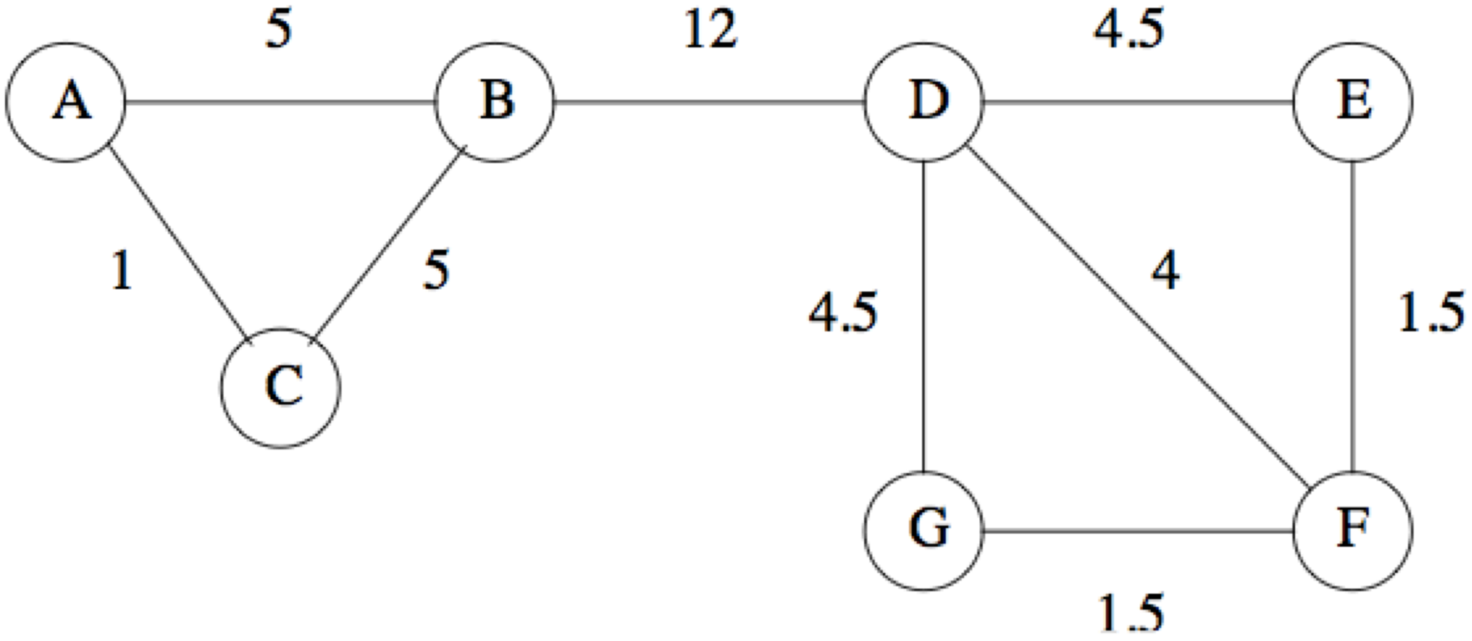


Figure 10.7: Betweenness scores for the graph of Fig. 10.1

Using Betweenness to Find Communities (3): Thresholding

- Could continue to remove edges with highest betweenness

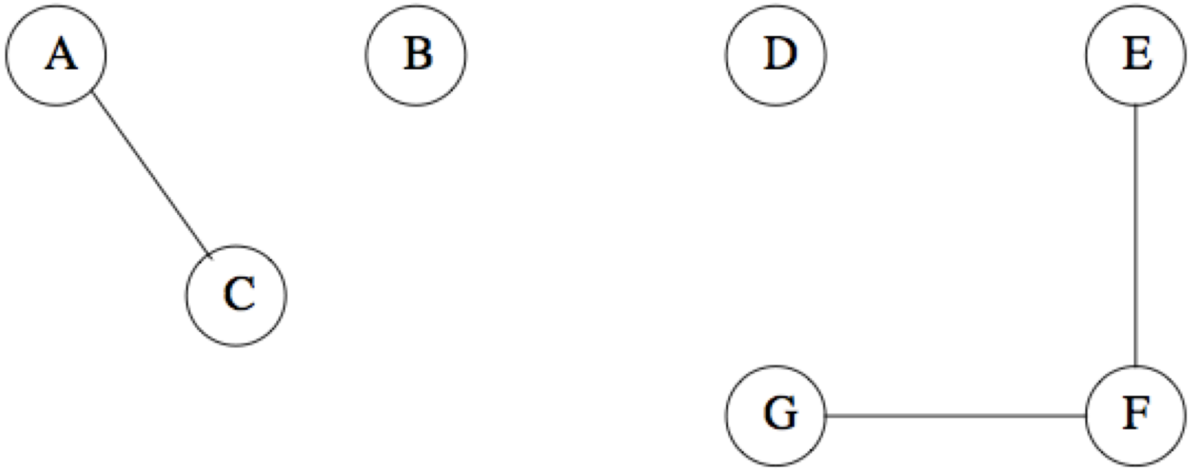


Figure 10.8: All the edges with betweenness 4 or more have been removed

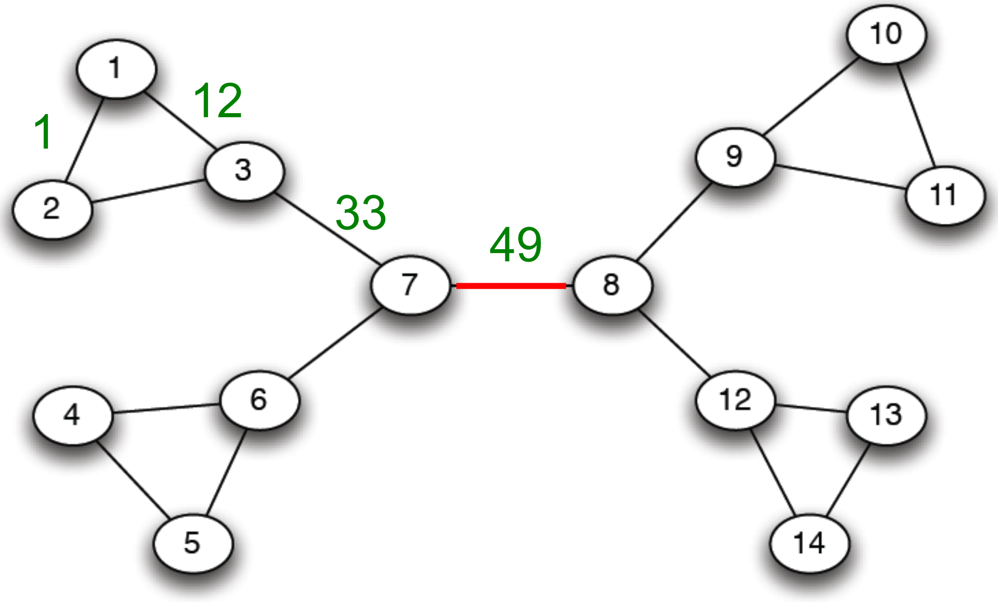
Run Girvan-Newman Iteratively for Community Detection

- Recall: Divisive hierarchical clustering based on the notion of edge **betweenness**:

  Number of shortest paths passing through the edge

- **Girvan-Newman Algorithm:**
  - **Undirected unweighted networks**
  - **Repeat until no edges are left:**
    - Calculate betweenness of edges
    - This time: remove edges with highest betweenness
  - Connected components are communities
  - Gives a hierarchical decomposition of the network

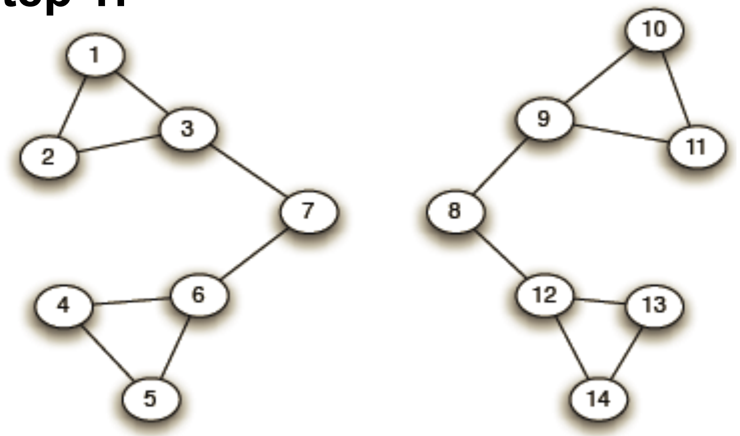
Girvan-Newman: Example



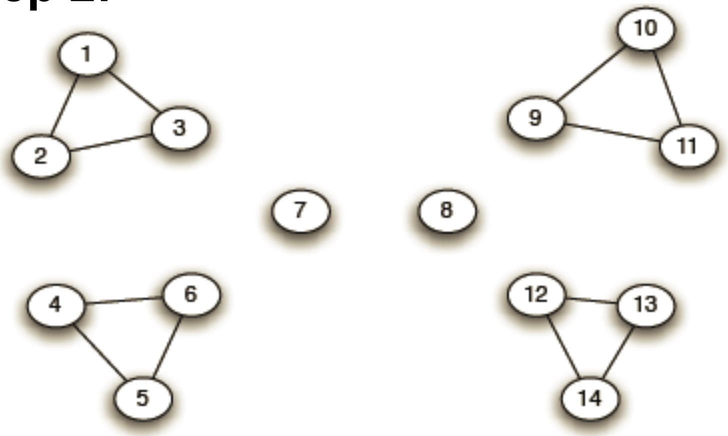
Need to re-compute betweenness at every step

Girvan-Newman: Example

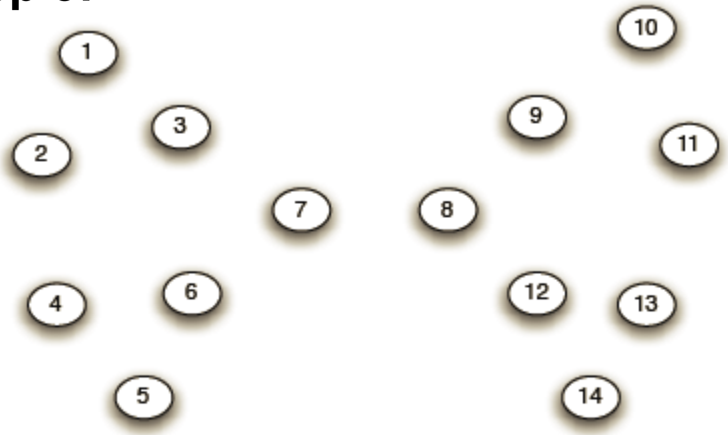
Step 1:



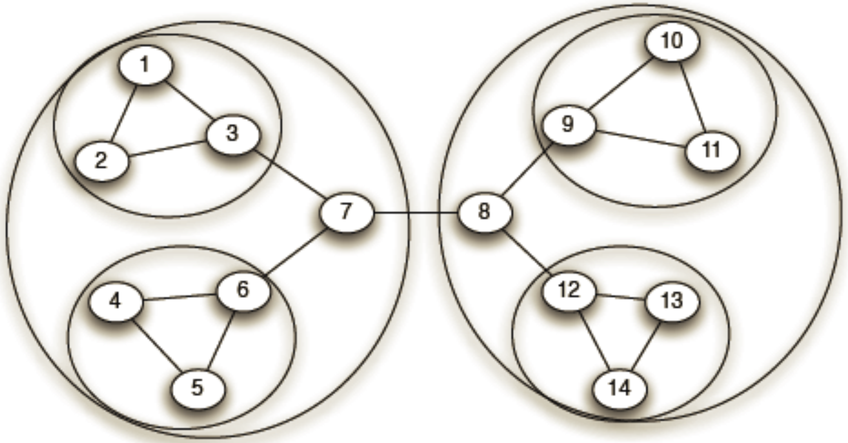
Step 2:



Step 3:

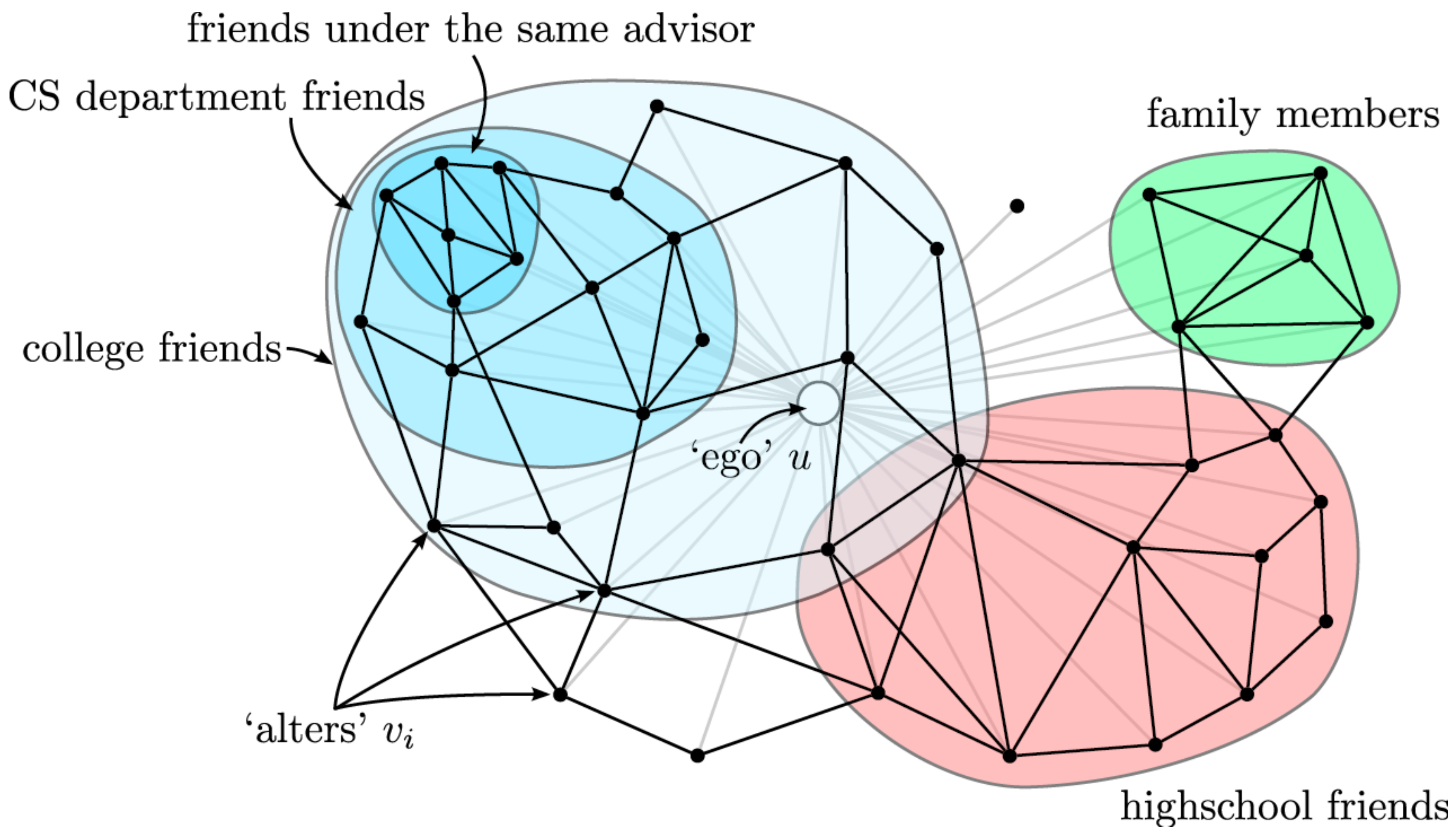


Hierarchical network decomposition:



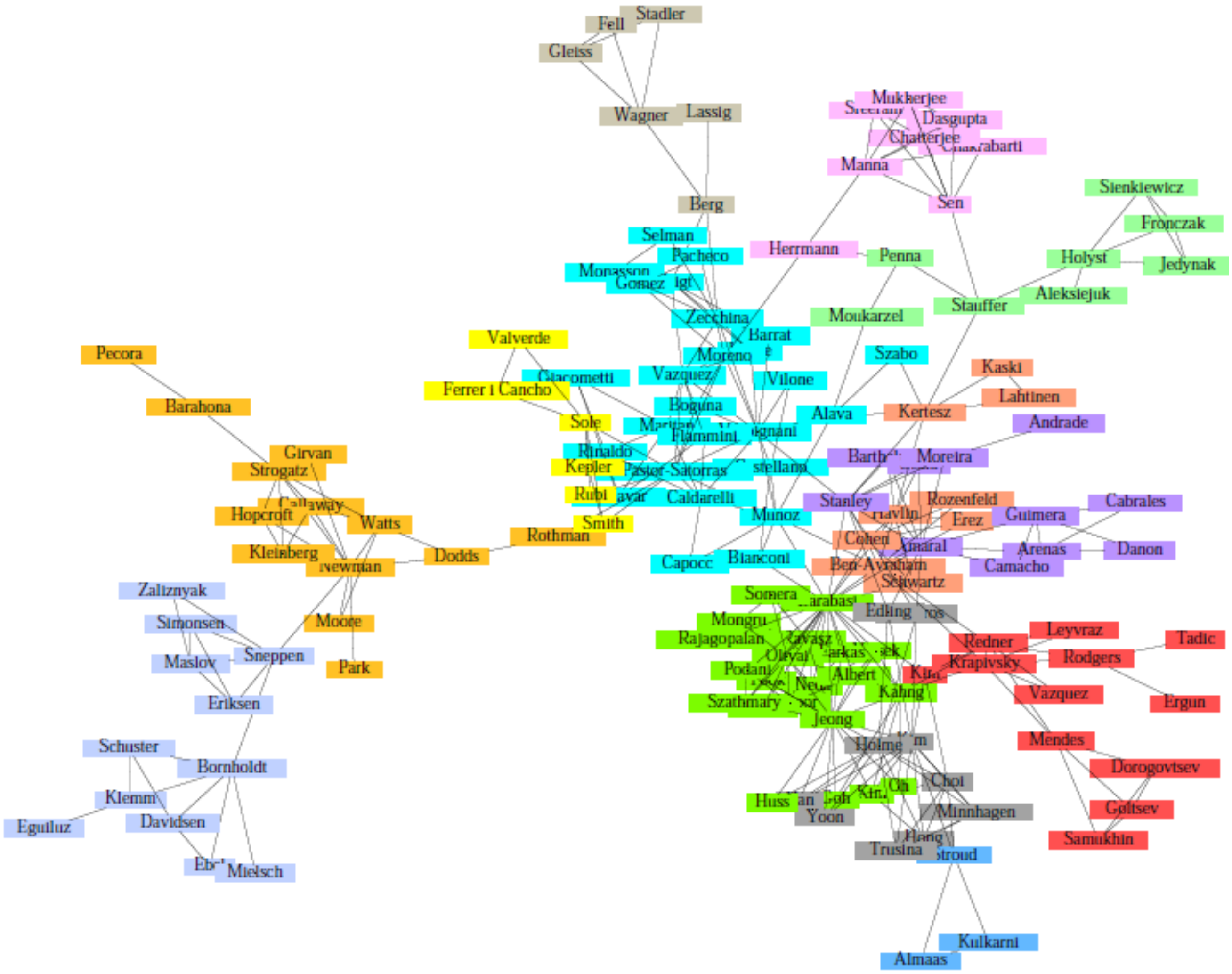
Recall: Twitter & Facebook

- **Discovering social circles, circles of trust:**



[McAuley, Leskovec: Discovering social circles in ego networks, 2012]

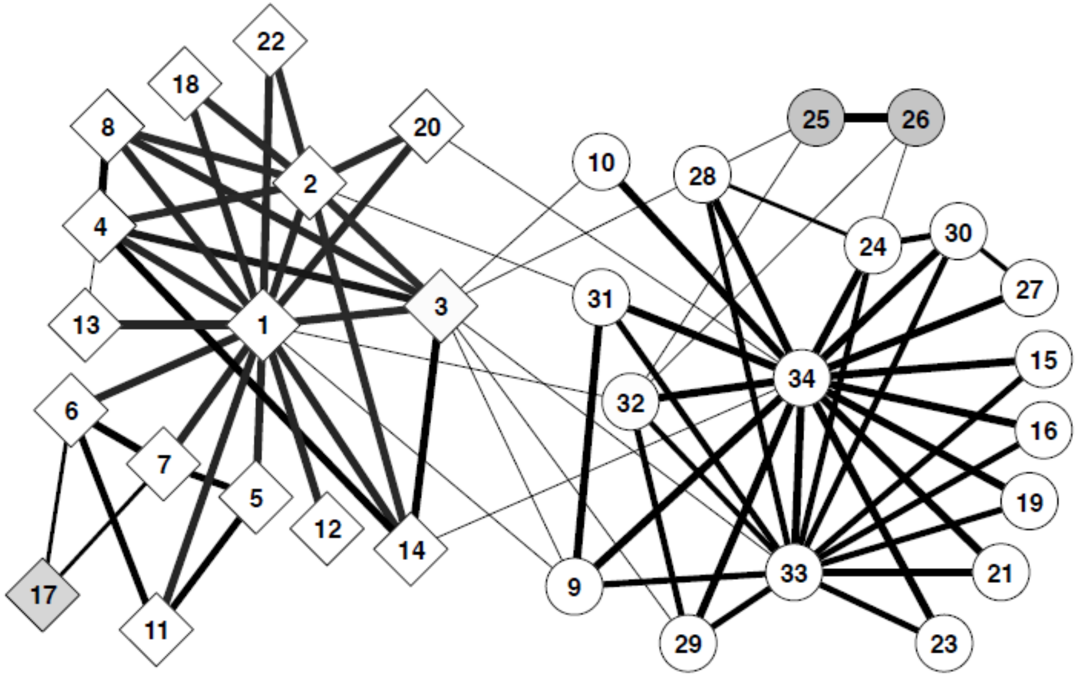
Girvan-Newman: Results

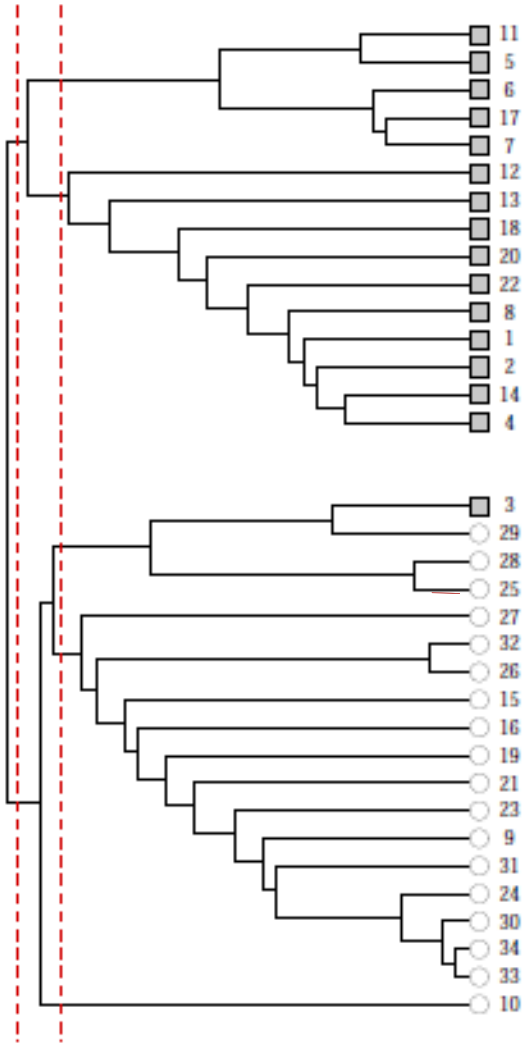


Communities in physics collaborations

Girvan-Newman: Results

- **Zachary's Karate club:** Hierarchical decomposition





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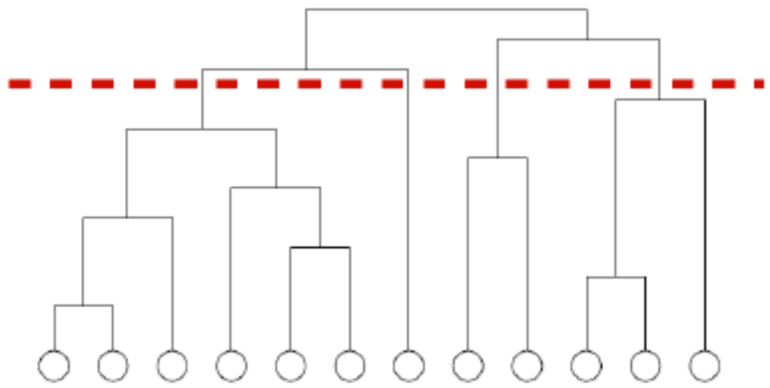
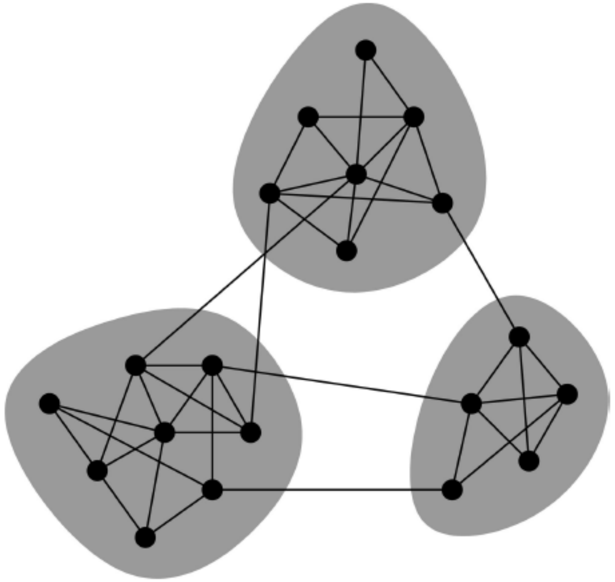


Network Communities

- **Communities:** sets of **tightly connected nodes**
- <u>Define:</u> **Modularity *Q***
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups $\in {}_{\in}\boldsymbol{S}$:

$$\boldsymbol{Q} = \underbrace{\sum_{s \in S} [\, (\#\ \text{edges within group } s) - (\text{expected } \#\ \text{edges within group } s)\,]}_{\text{Need a null model!}}$$



The null model is a graph which matches one specific graph in some of its structural features, but which is otherwise taken to be an instance of a random graph. The null model is used as a term of comparison, to verify whether the graph in question displays some feature, such as community structure, or not.

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Modularity

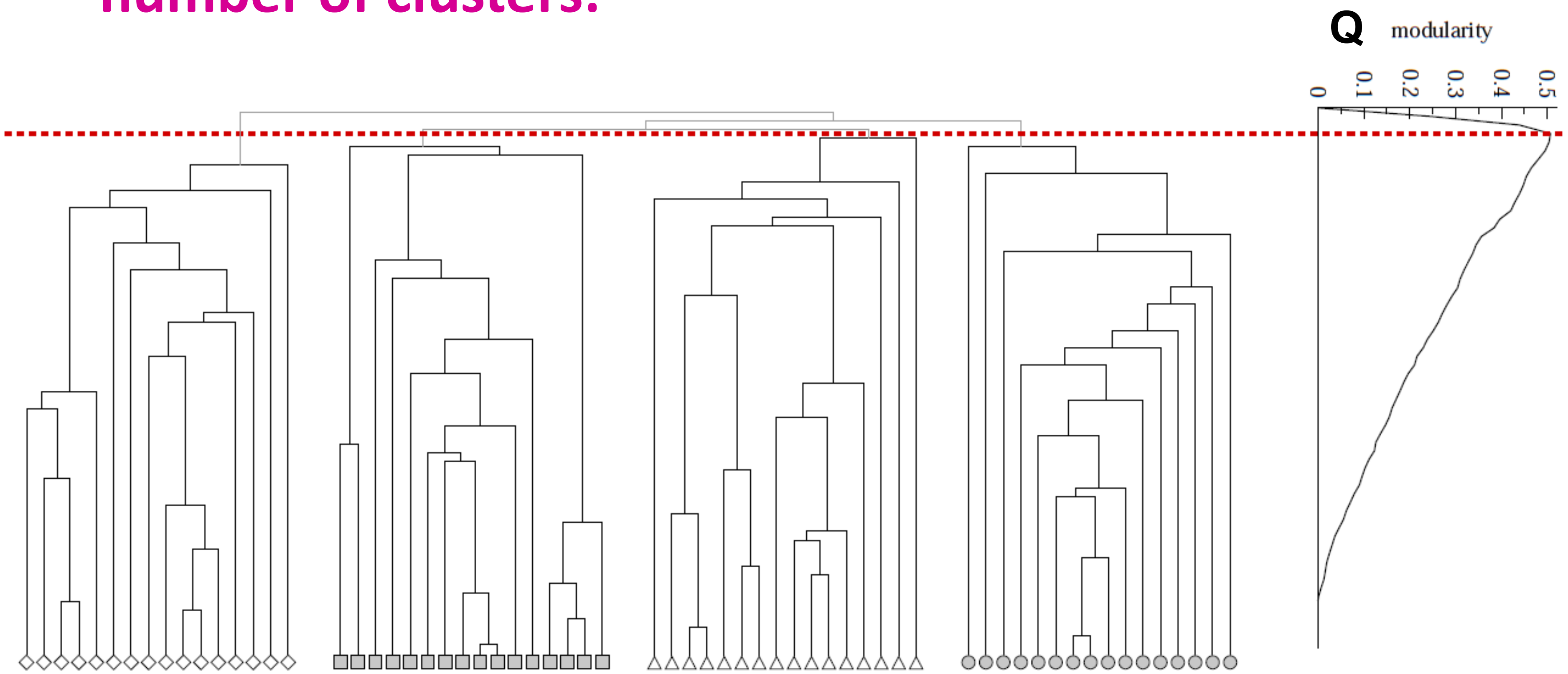
- **Modularity of partitioning S of graph G:**
  - $\mathbf{Q = \sum_{s \in S} [\ (\#\ edges\ within\ group\ \mathit{s}) -}$ $\mathbf{(expected\ \#\ edges\ within\ group\ \mathit{s})\ ]}$
  - $\boldsymbol{Q(G,S) = \underbrace{\frac{1}{2m}}_{\text{Normalizing cost.: } -1<Q<1} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left(A_{ij} - \frac{k_i k_j}{2m} \right)}$

    A_{ij} = 1 if i connects j,
    0 else

- **Modularity values take range [−1,1]**
  - It is positive if the number of edges within groups exceeds the expected number
  - **0.3-0.7<Q** means significant community structure

Modularity: Number of clusters

- **Modularity is useful for selecting the number of clusters:**



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Spectral Clustering

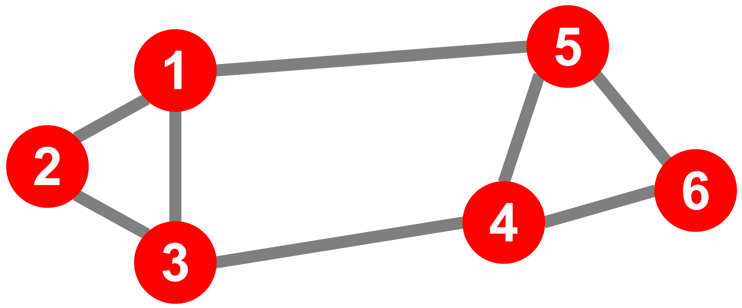
Another approach to organizing social-network graphs

Partitioning Graphs

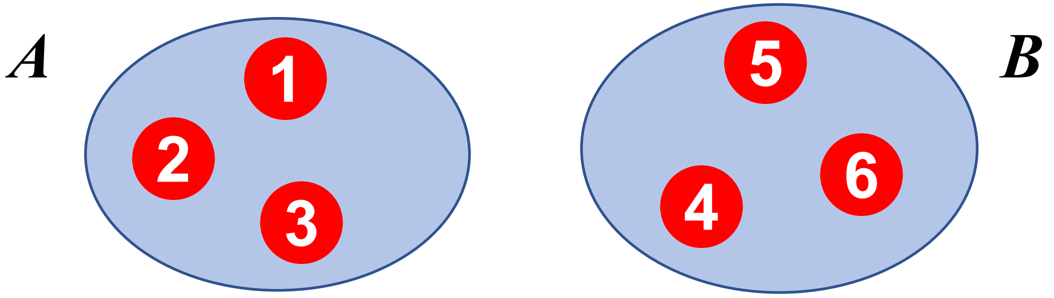
- Another approach to organizing social networking graphs
- Problem: partitioning a graph to minimize the number of edges that connect different components (communities)
- Goal of minimizing the cut size
- If you just joined Facebook with only one friend
  - Don't want to partition the graph with you disconnected from rest of the world
  - <u>Want components to be not too unequal in size</u>

Graph Partitioning

- **Undirected graph**



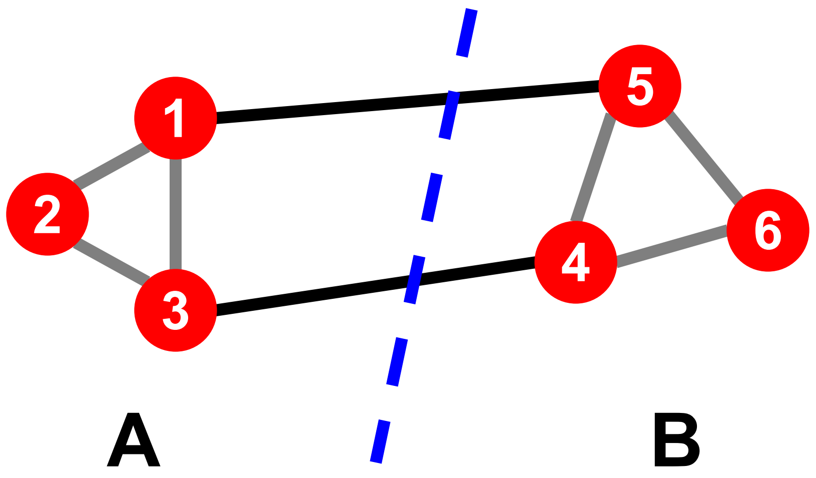
- **Bi-partitioning task:**
  - Divide vertices into two disjoint groups



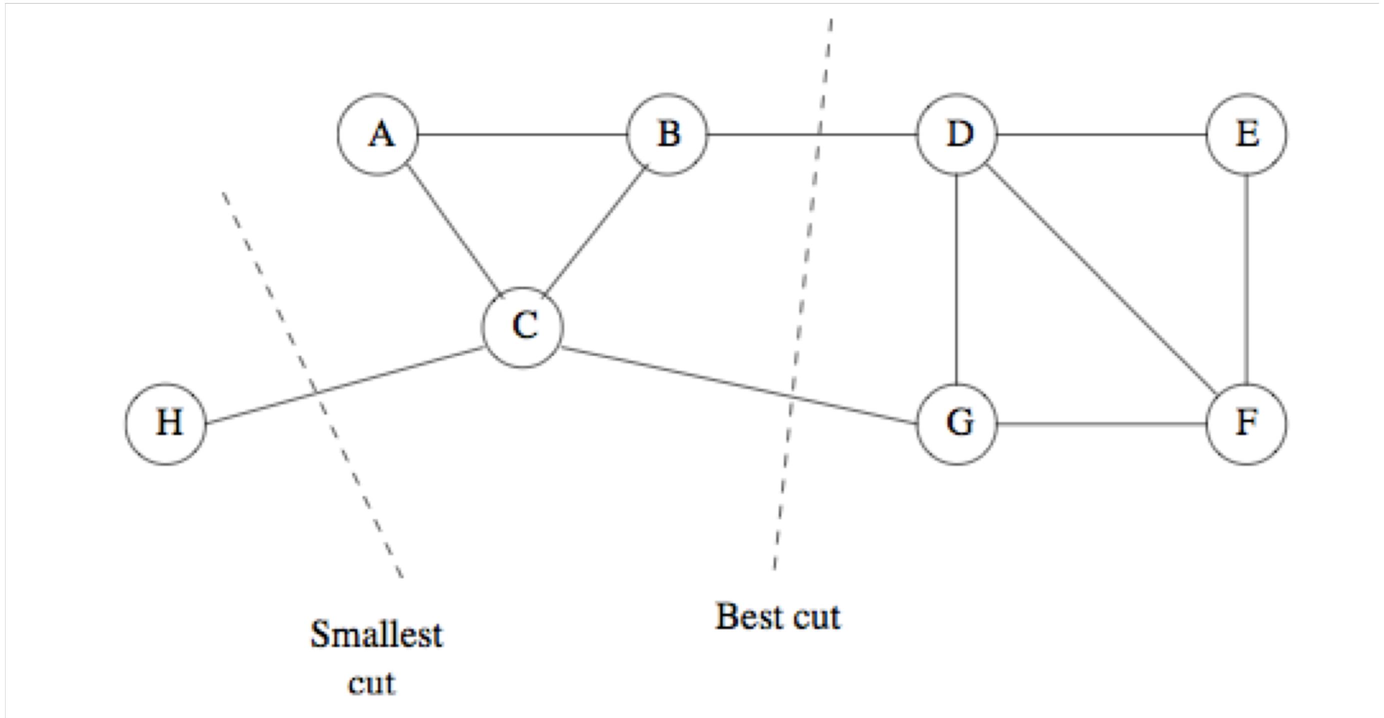
- **Questions:**
  - How can we define a “good” partition of ?
  - How can we efficiently identify such a partition?

Graph Partitioning

- **What makes a good partition?**
  - Divide nodes into two sets so that the **cut (set of edges that connect nodes in different sets) is minimized**
  - Want the two sets to be approximately equal in size
  - Maximize the number of within-group connections
  - Minimize the number of between-group connections



Example 10.14



- If we minimize cut: best choice is to put H in one set, other nodes in other set
- But: we reject partitions where one set is too small
- Better is to use cut with (B,D) and (C,G)
- **Smallest cut is not necessarily the best cut**

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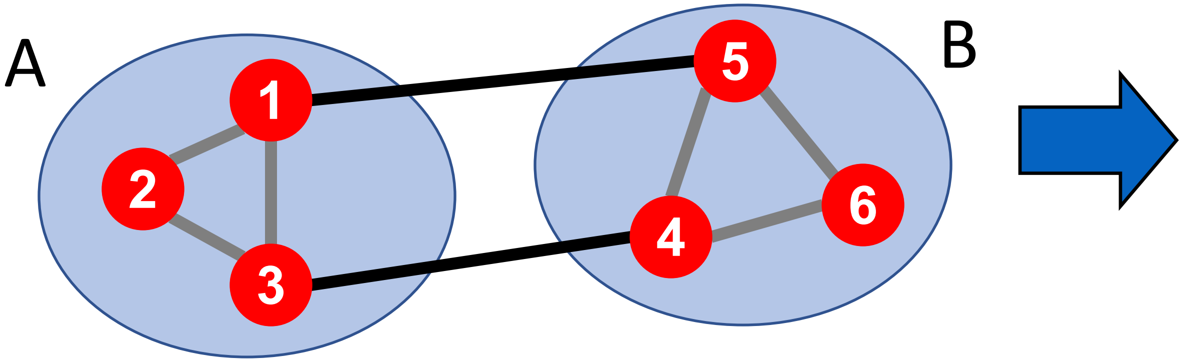
Graph Cuts

- **Express partitioning objectives as a function of the "edge cut" of the partition**
- **Cut:** Set of edges with only one vertex in a group:

$$cut(A,B) = \sum_{i \in A, j \in B} w_{ij}$$



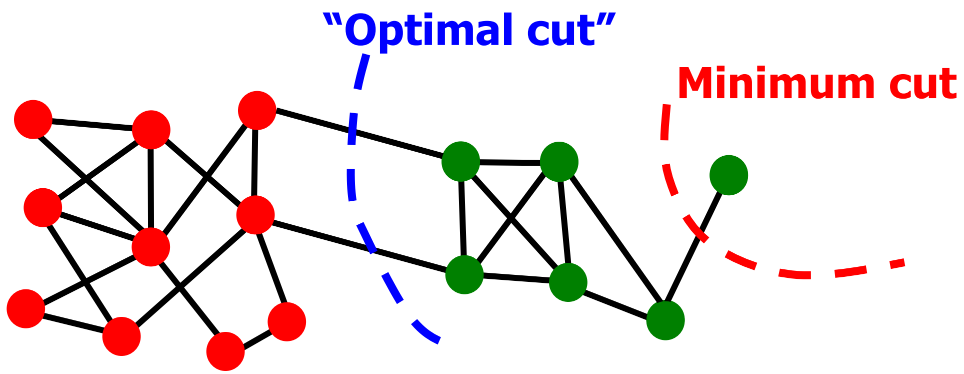
$cut(A,B) = 2$

Graph Cut Criterion

- **Criterion: Minimum-cut**
  - Minimize weight of connections between groups

$$\arg\min_{A,B} cut(A,B)$$

- **Degenerate case:**



- **Problem:**
  - Only considers external cluster connections
  - Does not consider internal cluster connectivity

Graph Cut Criteria

- **Criterion:** **Normalized-cut** [Shi-Malik, '97]
  - Connectivity between groups relative to the density of each group

$$ncut(A,B) = \frac{cut(A,B)}{vol(A)} + \frac{cut(A,B)}{vol(B)}$$

$\boldsymbol{vol(A)}$**:** total weight of the edges with at least one endpoint in $\boldsymbol{A}$: $\boldsymbol{vol(A) = \sum_{i \in A} k_i}$: $\boldsymbol{vol(A) = \sum_{i \in A} k_i}$

- **Why use this criterion?**
  - Produces more balanced partitions

- **How do we efficiently find a good partition?**
  - **Problem:** Computing optimal cut is NP-hard

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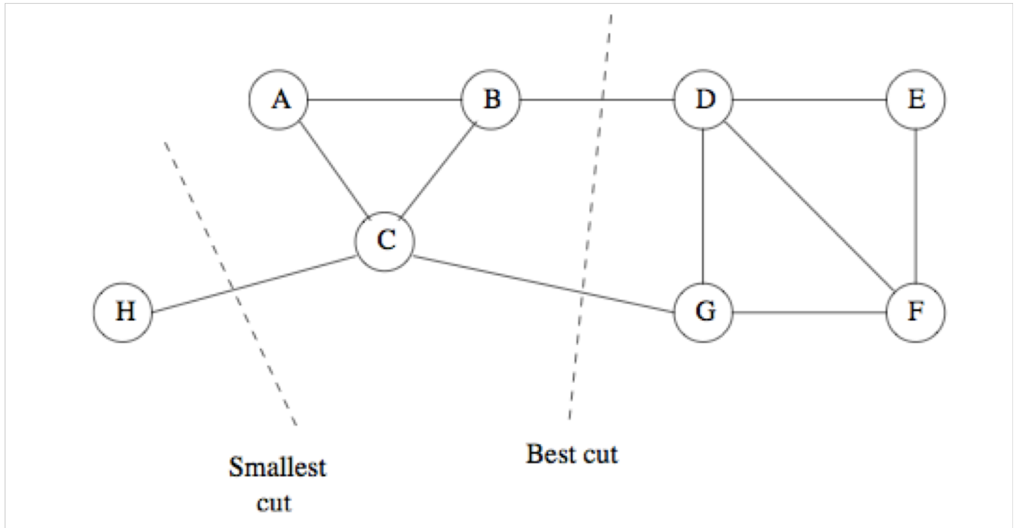
Example 10.15



- Partition nodes of graph into two disjoint sets S and T
- Normalized Cut for S and T is:

$$\frac{Cut(S,T)}{Vol(S)} + \frac{Cut(S,T)}{Vol(T)}$$

- If we choose S={H} and T={A,B,C,D,E,F,G} then Cut(S,T) = 1
  - Vol(S) = 1 (number of edges with at least one end in S)
  - Vol(T) = 11: all edges have at least one node in T
  - **Normalized cut is 1/1 + 1/11 = 1.09**
- For cut (B,D) and (C,G): S = {A,B,C,H}, T = {D,E,F,G}, Cut(S,T) = 2
- Vol(S) = 6, Vol(T) = 7, **normalized cut: 2/6 + 2/7 = 0.62**

Using Matrix Algebra to Find Good Graph Partitions

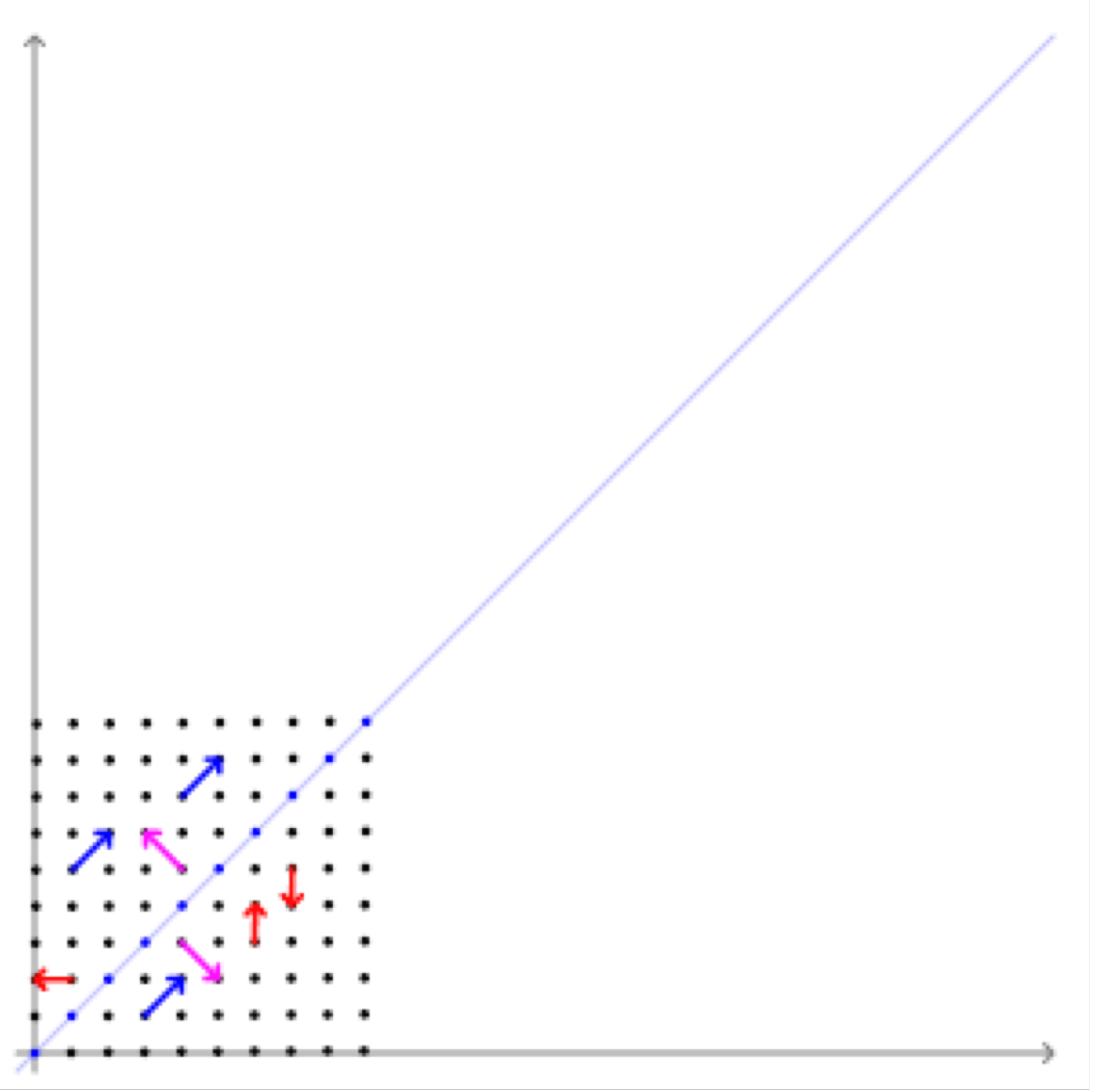
- Three matrices that describe aspects of a graph:
  - **Adjacency Matrix**
  - **Degree Matrix**
  - **Laplacian Matrix**: difference between degree and adjacency matrix

- Then get a good idea of how to partition graph from eigenvalues and eigenvectors of its Laplacian matrix

Recall: Eigenvalues and Eigenvectors

- The transformation matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ preserves the direction of vectors parallel to $\mathbf{v} = (1,-1)^T$ (in purple) and $\mathbf{w} = (1,1)^T$ (in blue). The vectors in red are not parallel to either eigenvector, so, their directions are changed by the transformation.

$$A\mathbf{v} = \lambda\mathbf{v}$$



http://setosa.io/ev/eigenvectors-and-eigenvalues/

https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

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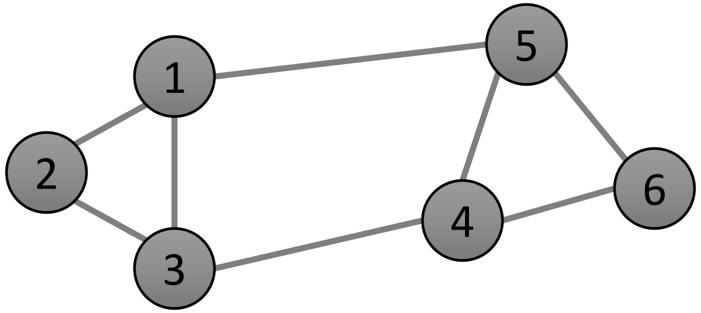
Matrix Representations

- **Adjacency matrix (A):**
  - $n \times n$ matrix
  - $A=[a_{ij}]$, $a_{ij}=1$ if edge between node i and j

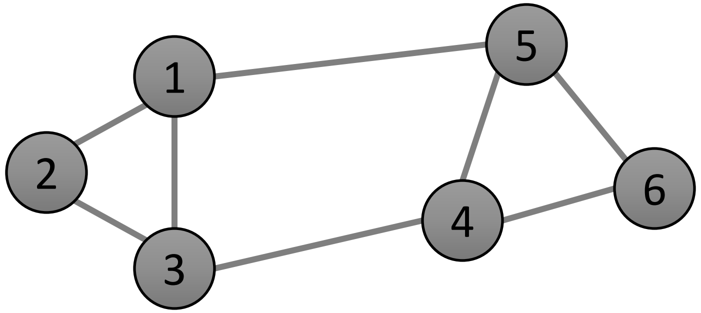


	1	2	3	4	5	6
1	0	1	1	0	1	0
2	1	0	1	0	0	0
3	1	1	0	1	0	0
4	0	0	1	0	1	1
5	1	0	0	1	0	1
6	0	0	0	1	1	0

- **Important properties:**
  - Symmetric matrix
  - Eigenvectors are real and orthogonal
    - dot_product(Eigenvectors_i, Eigenvectors_j) = 0

Matrix Representations

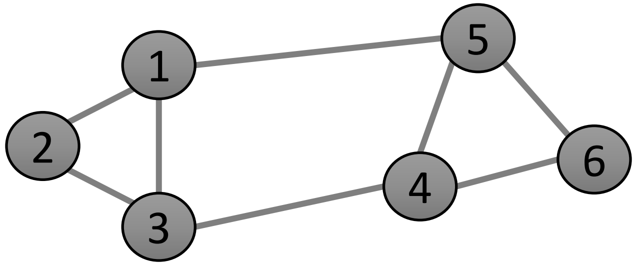
- **Degree matrix (D):**
  - $n \times n$ diagonal matrix
  - $D=[d_{ii}]$, d_{ii} = degree of node i



	1	2	3	4	5	6
1	3	0	0	0	0	0
2	0	2	0	0	0	0
3	0	0	3	0	0	0
4	0	0	0	3	0	0
5	0	0	0	0	3	0
6	0	0	0	0	0	2

Matrix Representations

- **Laplacian matrix (L):**
  - $\boldsymbol{n\ x\ n}$ symmetric matrix



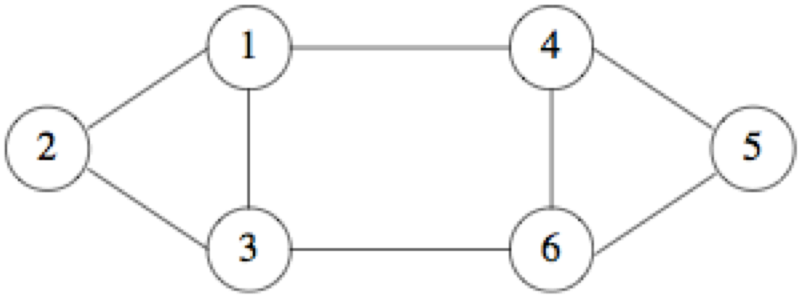
	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

$$\boldsymbol{L = D - A}$$

- **What is trivial eigenpair?**
  - $\boldsymbol{x} = (\mathbf{1}, \dots, \mathbf{1})$ then $\boldsymbol{L} \cdot \boldsymbol{x} = \mathbf{0}$ and so $\boldsymbol{\lambda} = \boldsymbol{\lambda_1} = \mathbf{0}$ **(smallest eigenvalue)**
- **Important properties of symmetric matrices:**
  - **Eigenvalues** are non-negative real numbers
  - **Eigenvectors** are real and orthogonal

$$\mathbf{x}^{\mathrm{T}}\mathbf{1} = \sum_{i=1}^{n} x_i = 0$$

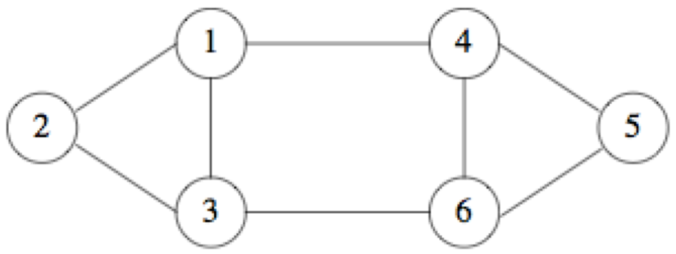
Example 10.19



$$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

- Graph and its Laplacian matrix

Example (cont.)



Eigenvalue	0	1	3	3	4	5
Eigenvector	1	1	−5	−1	−1	−1
	1	2	4	−2	1	0
	1	1	1	3	−1	1
	1	−1	−5	−1	1	1
	1	−2	4	−2	−1	0
	1	−1	1	3	1	−1

- Use standard math package to find all eigenvalues and eigenvectors
  - (Have not scaled eigenvectors to length 1, but could)
- Second eigenvector has three positive and three negative components
- Suggest obvious partitioning of {1,2,3} and {4,5,6}

λ_2 as optimization problem

- Recall: to find **second-smallest eigenvalue** for symmetric matrix (such as Laplacian):
  - Second smallest eigenvalue is the minimum of $x^T L x$ where $x = [x_1, x_2, \ldots, x_n]$ is a column vector (n x 1, n=# of nodes) (Rayleigh quotient)
  - Sum of $x_i^2 = 1$
  - x is orthogonal to the eigenvector associated with smallest eigenvalue
- **What is the meaning of min $\boldsymbol{x^{\mathrm{T}} L x}$ on *G*?**
  - $\mathrm{x^T L\, x} = \sum_{i,j=1}^{n} L_{ij}\, x_i x_j = \sum_{i,j=1}^{n} \left(D_{ij} - A_{ij}\right) x_i x_j$
  - $= \sum_i d_i x_i^2 - \sum_{(i,j)\in E} 2 x_i x_j$
  - $= \sum_{(i,j)\in E} (\underbrace{x_i^2 + x_j^2}) - 2x_i x_j) = \overset{\text{for each edge (i, j)}}{\sum_{(i,j)\in E} \left(x_i - x_j\right)^2}$

Node i has degree d_i. So, value x_i^2 needs to be summed up d_i times.
But each edge (i, j) has two endpoints so we need $x_i^2 + x_j^2$

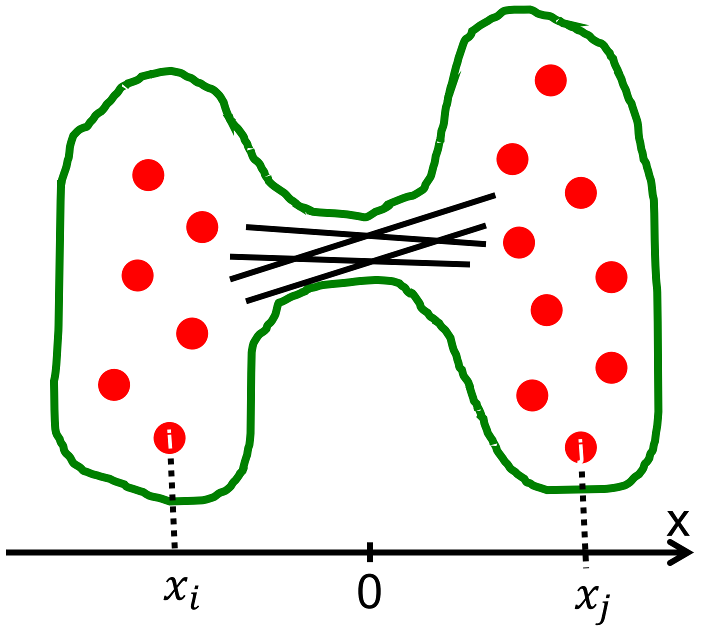
λ_2 as optimization problem (cont'd)

- **What else do we know about x?**
  - $\boldsymbol{x}$ is unit vector: $\sum_i \boldsymbol{x}_i^2 = \mathbf{1}$
  - $\boldsymbol{x}$ is orthogonal to $\mathbf{1}^{st}$ eigenvector $(\mathbf{1}, \dots, \mathbf{1})$ thus: $\sum_i \boldsymbol{x}_i \cdot \mathbf{1} = \sum_i \boldsymbol{x}_i = \mathbf{0}$

- **Remember:**

$$\lambda_2 = \min_{\substack{\text{All labelings} \\ \text{of nodes } i \text{ so} \\ \text{that } \sum x_i = 0}} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2}$$

We want to assign values x_i to nodes i such that few edges cross 0.
(we want x_i and x_j to subtract each other)



Balance to minimize

Ncut as an optimization problem

$$min_{\boldsymbol{x}} Ncut(\boldsymbol{x}) = min_{y} \frac{y^T(\mathbf{D}-\mathbf{W})\boldsymbol{y}}{\boldsymbol{y}^T\mathbf{D}\boldsymbol{y}},$$

$$\boldsymbol{y} = (\mathbf{1}+\boldsymbol{x}) - b(\mathbf{1}-\boldsymbol{x}), \quad b = \frac{k}{1-k}.$$

Given a partition of nodes of a graph, V, into two sets A and B, let $\boldsymbol{x}$ be an $N = |\boldsymbol{V}|$ dimensional indicator vector, $x_i = 1$ if node i is in A and -1, otherwise. Let $\boldsymbol{d}(i) = \sum_j w(i,j)$ be the

Now, recall a simple fact about the *Rayleigh quotient* [11]:

Let $\mathbf{A}$ be a real symmetric matrix. Under the constraint that $\boldsymbol{x}$ is orthogonal to the j-1 smallest eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}$, the quotient $\frac{\boldsymbol{x}^T\mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}$ is minimized by the next smallest eigenvector $\boldsymbol{x}_j$ and its minimum value is the corresponding eigenvalue λ_j.

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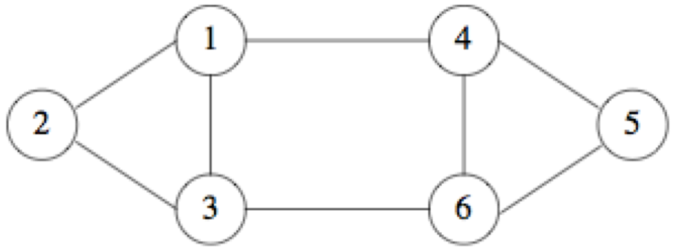
Recall: Example



Eigenvalue	0	1	3	3	4	5
Eigenvector	1	1	-5	-1	-1	-1
	1	2	4	-2	1	0
	1	1	1	3	-1	1
	1	-1	-5	-1	1	1
	1	-2	4	-2	-1	0
	1	-1	1	3	1	-1

- Use standard math package to find all eigenvalues and eigenvectors
    - (Have not scaled eigenvectors to length 1, but could)
- Second eigenvector has three positive and three negative components
- Suggest obvious partitioning of {1,2,3} and {4,5,6}

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So far…

- **How to define a "good" partition of a graph?**
  - Minimize a given graph cut criterion

- **How to efficiently identify such a partition?**
  - Approximate using information provided by the eigenvalues and eigenvectors of a graph

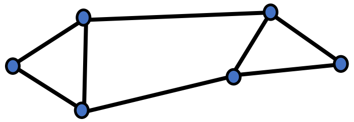
- **Spectral Clustering**
  - Naïve approache:
    - Split at **0**

Spectral Clustering Algorithms

- **Three basic stages:**
  - **1) Pre-processing**
    - Construct a matrix representation of the graph
  - **2) Decomposition**
    - Compute eigenvalues and eigenvectors of the matrix
    - Map each point to a lower-dimensional representation based on one or more eigenvectors
  - **3) Grouping**
    - Assign points to two or more clusters, based on the new representation

Spectral Partitioning Algorithm

- 1) **Pre-processing:**
  - Build Laplacian matrix L of the graph



	1	2	3	4	5	6
1	3	-1	-1	0	-1	0
2	-1	2	-1	0	0	0
3	-1	-1	3	-1	0	0
4	0	0	-1	3	-1	-1
5	-1	0	0	-1	3	-1
6	0	0	0	-1	-1	2

- 2) **Decomposition:**
  - Find eigenvalues λ and eigenvectors x of the matrix L

λ=

0.0
1.0
3.0
3.0
4.0
5.0

X =

0.4	0.3	-0.5	-0.2	-0.4	-0.5
0.4	0.6	0.4	-0.4	0.4	0.0
0.4	0.3	0.1	0.6	-0.4	0.5
0.4	-0.3	0.1	0.6	0.4	-0.5
0.4	-0.3	-0.5	-0.2	0.4	0.5
0.4	-0.6	0.4	-0.4	-0.4	0.0

  - Map vertices to corresponding components of λ_2

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

How do we now find the clusters?

Spectral Partitioning

- **3) Grouping:**
  - Sort components of reduced 1-dimensional vector
  - Identify clusters by splitting the sorted vector in two
- **How to choose a splitting point?**
  - Naïve approaches:
    - Split at **0** or median value
  - More expensive approaches:
    - Attempt to minimize normalized cut in 1-dimension (sweep over ordering of nodes induced by the eigenvector)

1	0.3
2	0.6
3	0.3
4	-0.3
5	-0.3
6	-0.6

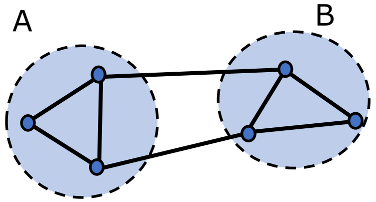
Split at 0:

Cluster A: Positive points

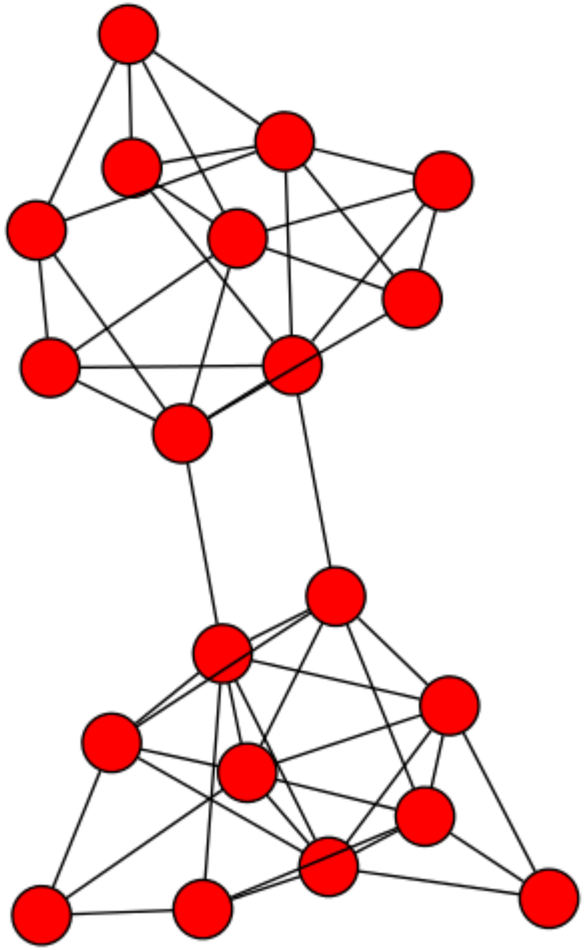
Cluster B: Negative points

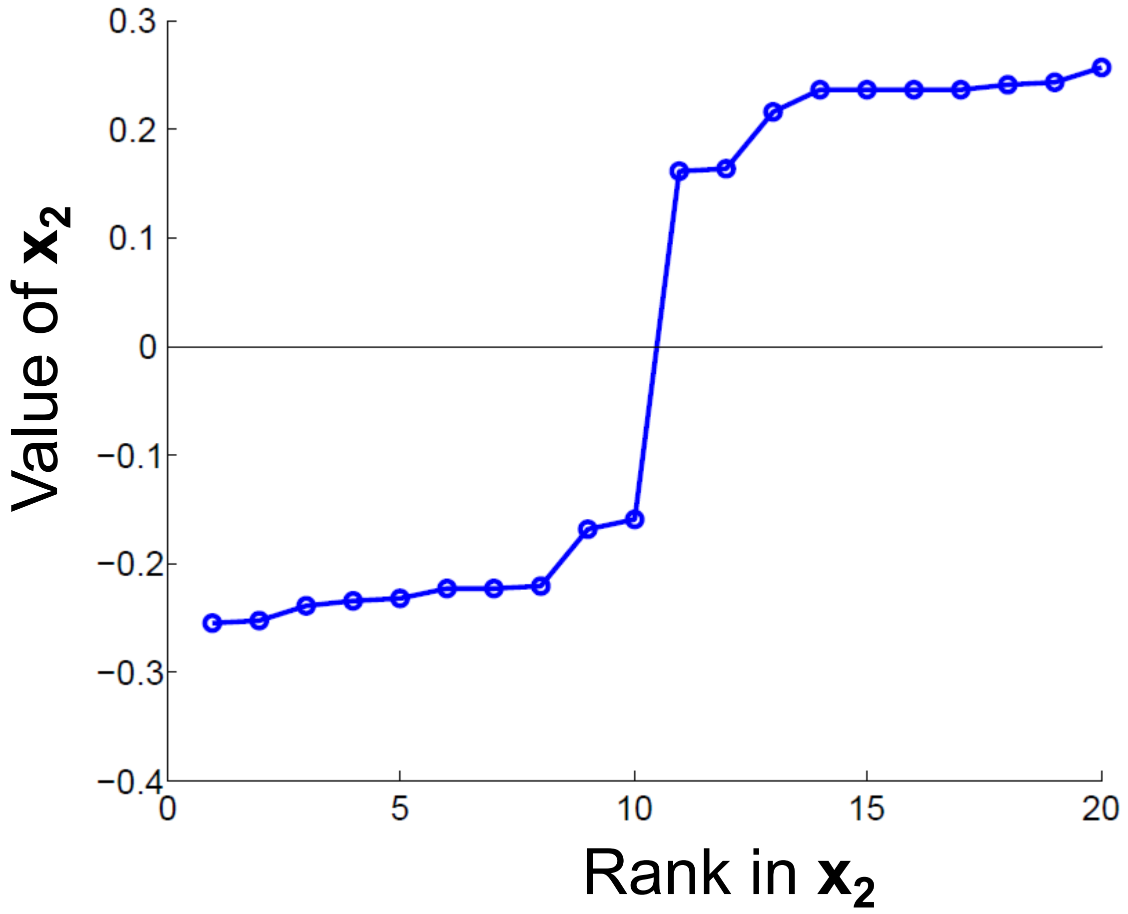
1	0.3
2	0.6
3	0.3

4	-0.3
5	-0.3
6	-0.6

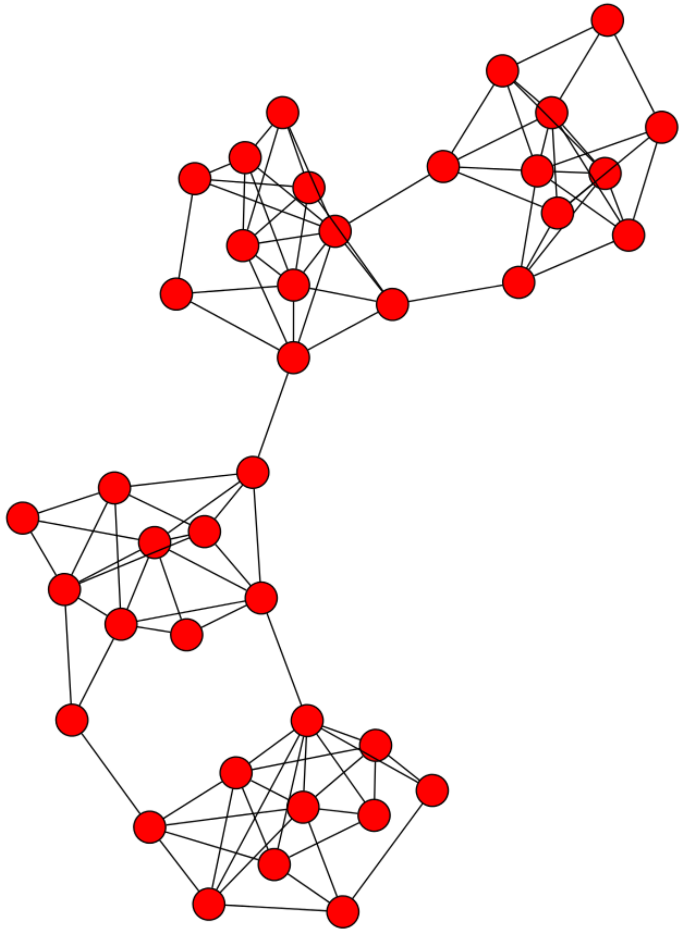


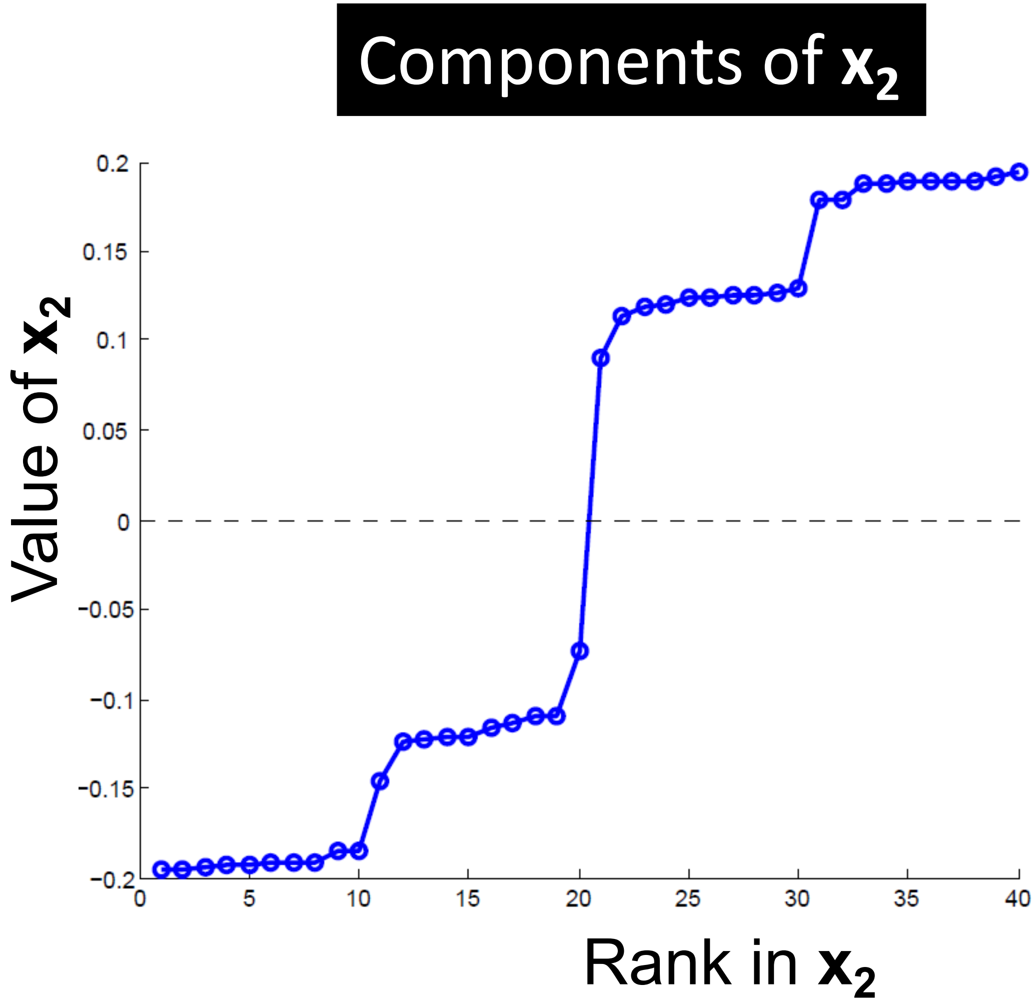
Example: Spectral Partitioning



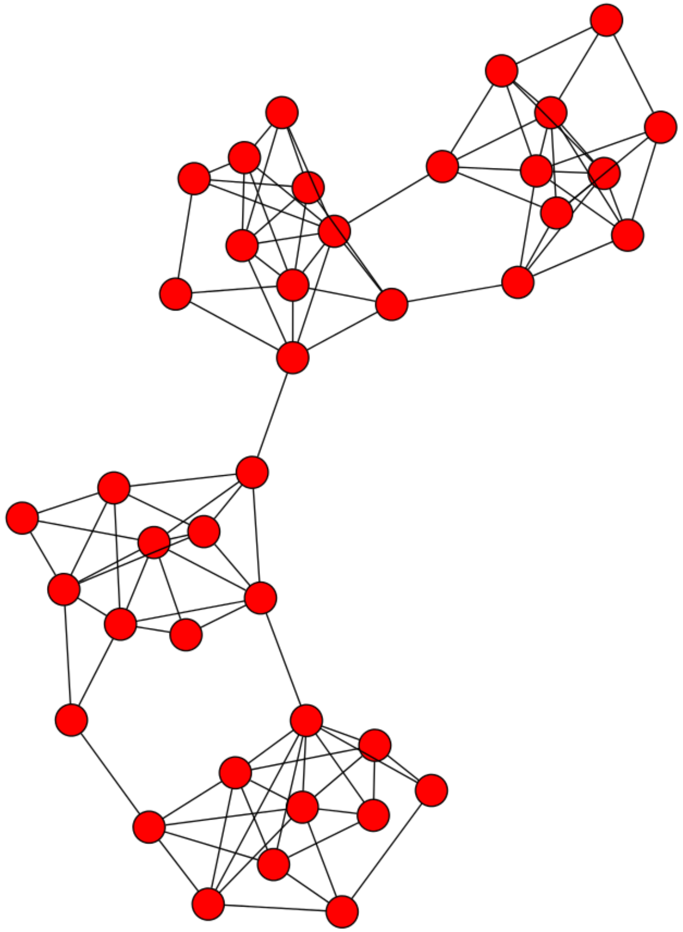


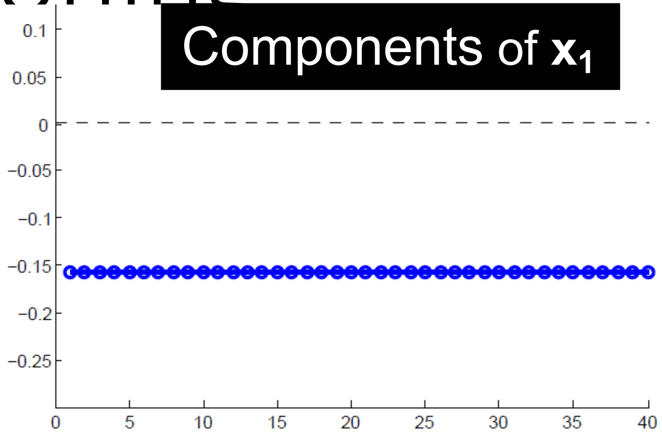
Example: Spectral Partitioning

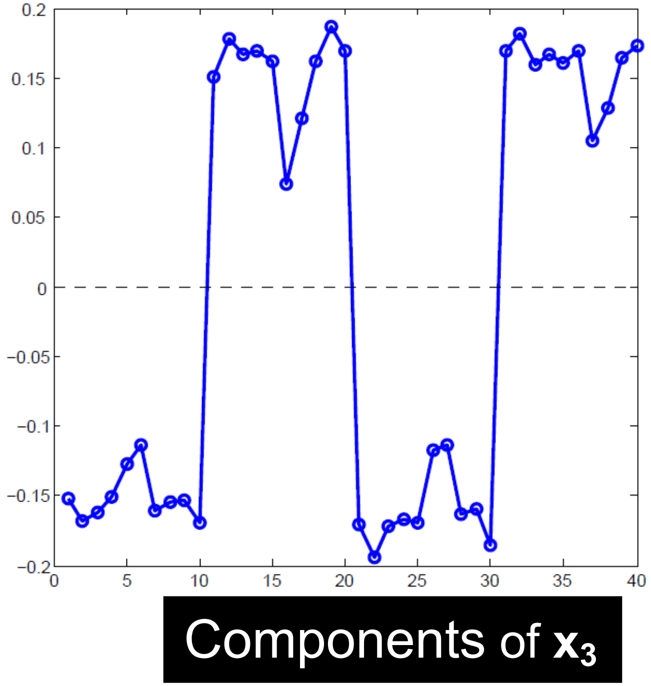




Example: Spectral partitioning







k-Way Spectral Clustering

- **How do we partition a graph into k clusters?**

- **Two basic approaches:**
    - **Recursive bi-partitioning** [Hagen et al., ’92]
        - Recursively apply bi-partitioning algorithm in a hierarchical divisive manner
        - Disadvantages: Inefficient, unstable
    - **Cluster multiple eigenvectors** [Shi-Malik, ’00]
        - Build a reduced space from multiple eigenvectors
        - Commonly used in recent papers
        - Multiple eigenvectors prevent instability due to information loss
        - A preferable approach…